UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE

Applied Mathematics Honours	**APK Campus**
APM8X03 Relativity A	
03 JUNE 2019	

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TIME 3 Hours **MARKS** 100

Please read the following instructions carefully

1. Answer all the questions.
2. Answer each of the three questions in a separate booklet.
3. You may use a calculator.
4. This paper consists of 4 pages, including this one.
5. Some useful results are listed on the last page of this paper and may be used without proof.

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QUESTION 1

(a) In $\mathbb{R}^2$ a basis transformation is made from

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \text{ to } \left\{ \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}.$$

Find the transition matrix. (8)

(b) The paraboloidal coordinates (u, v, ϕ) are defined by

$$x = uv\cos\phi, \quad y = uv\sin\phi, \quad z = \frac{1}{2}(u^2 - v^2);$$

where

$$u \geq 0, \quad v \geq 0, \quad 0 \leq \phi < 2\pi.$$

(i) Find the equations for a ϕ curve and describe its geometry. (4)

(ii) Derive the tangent basis vectors for the coordinate system. (3)

(iii) Derive the metric tensor for the coordinate system. (6)

(iv) Derive the conjugate metric tensor for the coordinate system. (4)

(c) T^k are the components of a contravariant tensor of rank 1 and g_{ij} are the components of a covariant tensor of rank 2. Show that $g_{ij}T^j$ are the components of a covariant tensor of rank 1. (4)

(d) $\mathbf{T}$ is a tensor of rank 2 in $\mathbb{R}^N$. Show that $T_{(ij)} - \dfrac{1}{N} T^k_{\ k} \, g_{ij}$ is traceless, where the notation $T_{(ij)}$ denotes a symmetrised tensor. (6)

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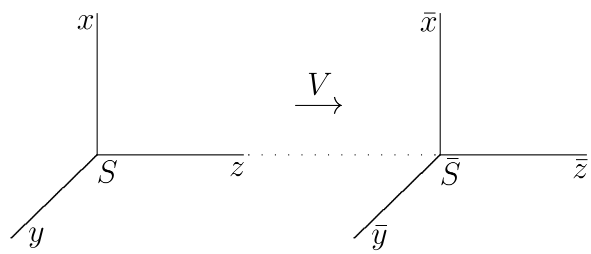
QUESTION 2

(a) In the figure two inertial systems S and $\bar{S}$ are shown. $\bar{S}$ moves at a constant speed V with respect to S. The coordinate systems are chosen such that the relative direction of motion is the z-direction in both frames.



The space and time coordinates of two events are measured in the S frame as follows:

Event A	$x_{\mathrm{A}} = y_{\mathrm{A}} = 0$	$z_{\mathrm{A}} = 2a$	$t_{\mathrm{A}} = a/c$
Event B	$x_{\mathrm{B}} = y_{\mathrm{B}} = 0$	$z_{\mathrm{B}} = 3a$	$t_{\mathrm{B}} = 5a/(4c)$

where c is the speed of light and a is a constant. These two events are instantaneous in the $\bar{S}$ frame.

(i) Draw a Minkowski diagram of frames S and $\bar{S}$ and indicate events A and B on your diagram. (4)

(ii) Use the Lorentz transformation to calculate the speed V of the $\bar{S}$ frame relative to S. (6)

(iii) At what time do these events occur in the $\bar{S}$ frame and where do they occur in this frame? (5)

(b) $\bar{S}$ moves at a constant speed V_1 relative to S and $\bar{\bar{S}}$ moves in the same direction at constant speed V_2 relative to $\bar{S}$.

(i) Derive the transformation $S \to \bar{\bar{S}}$. What conclusion can you make about the speed V of $\bar{\bar{S}}$ relative to S? [Hint: Make use of the fact that the Lorentz transformation can be written in matrix form as $\bar{\mathbf{x}} = L\mathbf{x}$.] (13)

(ii) Show that there is an upper limit to V and find this limit. (7)

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QUESTION 3

(a) The Minkowski metric is given by

$$(\eta_{\mu\nu}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Write the line element ds in Minkowski space as a multiple of the time element dt by introducing the quantities $\beta_v = \dfrac{v}{c}$, $\gamma_v = \dfrac{1}{\sqrt{1-\beta_v^2}}$. (3)

(b) The velocity components of a moving particle in the S system in Question 2(a) is given by $v^k = \dfrac{dx^k}{dt}$ and in the $\bar{S}$ system by $\bar{v}^k = \dfrac{d\bar{x}^k}{d\bar{t}}$. Define a four-velocity in Minkowski space and make use of it in order to derive the velocity transformation

$$\bar{v}^{1,2} = \frac{Q}{\gamma} v^{1,2}, \quad \bar{v}^3 = Q(v^3 - V), \quad Q = \frac{1}{1 - \frac{Vv^3}{c^2}}. \qquad (13)$$

(c) An electron e^- has a kinetic energy of $\frac{9}{10}\, m_e c^2$, where m_e is the *rest mass* of the electron and c is the speed of light. It makes a head-on collision with a positron e^+ that is at rest. (A positron has the same mass as an electron but opposite charge.) In the collision the two particles annihilate each other and produce two photons γ of equal energy. The reaction can be written as

$$e^- + e^+ \to 2\gamma.$$

Answer the following questions, giving all results in terms of m_e and c.

(i) Determine the speed at which the electron was travelling before it collided with the positron. (6)

(ii) What is the momentum of the electron? (2)

(iii) What is the total energy of the electron? (2)

(iv) What is the total energy of the positron? (3)

(v) What is the energy of each photon? (3)

(vi) What is the momentum of each photon? (3)

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Useful Results

Lorentz transformation

$$\beta := \frac{V}{c},\ \gamma := \frac{1}{\sqrt{1-\beta^2}}$$

$$\bar{x}^1 = x^1,\ \bar{x}^2 = x^2,\ \bar{x}^3 = \gamma(x^3 - \beta\, x^4),\ \bar{x}^4 = \gamma(-\beta\, x^3 + x^4)$$

$$\bar{x} = x,\ \bar{y} = y,\ \ \bar{z} = \gamma(z - \beta\, ct),\ \ \bar{t} = \gamma\left(t - \tfrac{\beta}{c}\, z\right)$$

Relativistic Dynamics

$$\beta_v := \frac{v}{c},\ \gamma_v := \frac{1}{\sqrt{1-\beta_v^2}}$$

$$m = \gamma_v m_0$$

$$E\ =\ mc^2\ =\ T + m_0c^2$$

$$E^2 = p^2c^2 + m_0^2c^4$$

The photon has no rest mass.