UNIVERSITY OF JOHANNESBURG
FACULTY OF SCIENCE

MATHEMATICS (APK CAMPUS)

ASMA2A4

DISCRETE MATHEMATICS - IT

NOVEMBER 2018

EXAMINER: Ms. S. Richardson

MODERATOR: Ms C. Marais

2 HOURS 70 MARKS

DATE: 21/11/2018 TIME: 16:30

SURNAME AND INITIALS:____________________

STUDENT NUMBER:____________________

TELEPHONE NUMBER: ____________________

Instructions:

1. The paper consists of 10 printed pages, **excluding the front page.**
2. Answer all questions.
3. Write out all calculations (steps).
4. Read the questions carefully.
5. Questions are to be answered on the question paper in the space provided. If you do not have enough space to write an answer, complete it on the back of the previous page. Indicate this clearly.
6. Non-programmable calculators are allowed.
7. Good luck!

SECTION A: LOGIC 19 MARKS

1. Consider the language $\mathcal{L}_{\mathcal{H}}$ for humans, with function symbols m for 'the mother of', f for 'the father of', the unary predicate symbols M for 'man', W for 'woman', and the binary predicate symbols C with $\mathsf{C}(x, y)$ meaning 'x is a child of y', P with $\mathsf{P}(x, y)$ meaning 'x is a parent of y', K with $\mathsf{K}(x, y)$ meaning 'x knows y', and L with $\mathsf{L}(x, y)$ meaning 'x loves y'. The language also contains constant symbols Mary, John and Adam.

   Translate the $\mathcal{L}_{\mathcal{H}}$-formula into English. [3]

$$\exists x \exists y (\mathsf{W}(x) \wedge \mathsf{C}(x, y) \wedge \mathsf{C}(y, \mathsf{Mary}) \wedge (\mathsf{L}(\mathsf{John}, x) \vee \mathsf{L}(\mathsf{Adam}, x)))$$

2. Translate the $\mathcal{L}_{\mathcal{Z}}$-formula into English and determine if it is true in the structure of $\mathcal{Z}$:

$$\forall x (x > 0 \rightarrow \exists y \exists z (\neg y = z \wedge x = y + z))$$

[3]

3. Let P be a unary predicate symbol, and R be a binary predicate symbol.We will determine if

$$\forall x \exists y R(x, y),\ \forall x \forall y (R(x, y) \rightarrow P(y)) \models \exists y P(y).$$

   by answering the following questions:

   (a) Convert the following formula into prenex conjunctive normal form: [2]

$$\forall x \forall y (R(x, y) \rightarrow P(y))$$

(b) Skolemize the result you obtained in (a), as well as $\forall x \exists y R(x,y)$ and $\neg \exists y P(y)$. [2]

(c) Write your results in (b) in clausal form. [2]

(d) Use resolution to determine whether

$$\forall x \exists y R(x,y), \forall x \forall y (R(x,y) \to P(y)) \models \exists y P(y).$$

Remember to indicate the most general unifiers used. [3]

4. Use symantic tableaux to determine if the following is a tautology: [4]

$$(\neg q \wedge (p \to q)) \to \neg p$$

SECTION B: COMBINATORICS

21 MARKS

1. Prove that if $C(n, r) = C(n, s)$ and $r = s$, then $n = r + s$. [4]

2. Consider the letters in the word SCHMALTZY.

   (a) How many different arrangements of these letters are there? [1]

   (b) How many arrangements are there on which the S occurs as one of the first three letters? [3]

3. Determine the coefficient of x^5y^3 in the expansion of $(2x^2 - \frac{3y}{x})^7$. [3]

4. At a cafeteria customers compose a meal by choosing a sandwich, and beverage and a desert. There are 5 different types of sandwich, 9 different types of beverage, and 3 different types of desert.

   (a) How many different meals can be composed in this way? [1]

   (b) What is the least number of customers that have to order a meal so that we can be sure, without looking, that at least 3 customers have ordered exactly the same meal? [3]

5. In a sample of 50 *Netflix* subscribers, 15 like action movies and 18 like comedies. Nine like action and comedy, 10 like comedy and documentaries, 8 like action and documentaries. Seven of the subscribers like all three genres and 18 like none of the genres.

   (a) Draw a Venn diagram to illustrate the given information. [3]

(b) How many of the subscribers like at least one of the three genres? [1]

(c) How many like documentaries? [2]

Section C: Number Theory 30 Marks

1. Prove that the sum of an odd integer and an even integer is odd. [2]

2. Divide -100 by 13 with a remainder. [1]

3. Consider the set $\mathbb{Z}_8$ and answer the questions that follow:

   (a) Evaluate $[7]_8 \oplus_8 [8]_8$. [1]

   (b) Evaluate $[8]_8 \odot_8 [8]_8$. [1]

   (c) Find the additive inverse of $[3]_8$ in $\mathbb{Z}_8$. [1]

(d) Find the multiplicative inverse of $[4]_8$ in $\mathbb{Z}_8$ if it exists. If not, explain why it does not exist. [1]

4. Use the Euclidean algorithm to calculate $\gcd(822, 1998)$. [3]

5. Find $m, n \in \mathbb{Z}$ such that $822m + 1998n = \gcd(822, 1998)$. [3]

6. Consider the following two linear Diophantine equations:

$$\begin{aligned} 822x + 1998y &= 456 \\ 1998x + 822y &= 356 \end{aligned}$$

(a) Indicate which of the two has no integral solutions and explain why. [2]

(b) Determine the general solution of the equation that does have integral solutions. [2]

7. Use Fermat's Little Theorem to prove that 17 divides $11^{104} + 1$. [3]

8. If Jane's age is divided by 5, the remainder is 4. If it is divided by 3, the remainder is 2. If it is divided by 7, the remainder is 5. How old is Jane? [4]

9. Suppose Alice wants her friends to encrypt e-mail messages before sending them to her. Alice and her friends agreed to simply write 1 instead of A, 2 instead of B, and so on, up to 26 instead of Z. They will also write 0 instead of an empty space.

   If Alice's private key is $(221, 5)$, determine e in her corresponding public key $(221, e)$ for this RSA crypto-system. [3]

10. Prove that every natural number greater than one has a prime divisor. [3]