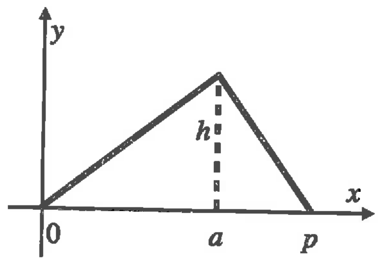
Department of Pure and Applied Mathematics
Differential Equations B (APM8X11) – Final test

14 November, 2018

Examiner: Prof. Fabio Cinti

Duration: 3 hours
Total marks: 50
This is a closed-book test
Note: numbers in brackets [ ] indicate the points that are awarded for the given part of the question.

1. Consider a string of length p and endpoints at $x = 0$ and $x = p$ depicted as follows:



Derive the function $f(x)$ that represents the shape of the string and calculate its sine series representation.

[7]

2. Use the d'Alembert's formula to solve the boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \; 0 < x < L, \; t > 0\,,$$

$u(0,t) = u(L,t) = 0$ (for all $t > 0$), $u(x,0) = \sin(\pi x)$ and $\frac{\partial u}{\partial t}(x,0) = 0$ (for $0 < x < L$ and c being a constant). Determine the first time the string returns to its initial shape.

[8]

3. Show that the solution of the heat equation:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \; t > 0$$

(c being a constant), with boundary conditions

$$u(0,t) = 0, \text{ and } u_x(L,t) = 0\,,$$

and initial conditions

$$u(x,0) = f(x)\,,$$

is

$$u(x,t) = \sum_{n=0}^{\infty} B_n \sin\left[\frac{\pi}{2L}(2n+1)x\right] \exp\left\{-\left[c\frac{\pi}{2L}(2n+1)\right]^2 t\right\}\,,$$

where

$$B_n = \frac{2}{L}\int_0^L dx\, f(x) \sin\left[\frac{\pi}{2L}(2n+1)x\right].$$

[15]

4. Compute the Laplacian in spherical coordinates of the function

$$u(x,y,z) = (x^2+y^2+z^2)^{3/2}$$

determine if u satisfies the Laplace's equation

$$\nabla^2 u = 0.$$

[5]

5. Consider a circular membrane with center in the origin of the axis and radius $a = 2$. Solve the vibrating membrane problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

where $u = u(r,t)$, $c = 1$, $0 < r < a$, $t > 0$, with boundary conditions

$$u(a,t) = 0\,, \; t > 0$$

and radially symmetric initial conditions

$$u(r,0) = 0\,, \; \frac{\partial u}{\partial t}(r,0) = 1, \; 0 < r < a.$$

[15]