UNIVERSITY
OF
JOHANNESBURG

FACULTY OF SCIENCE

DEPARTMENT:	**PURE AND APPLIED MATHEMATICS**
MODULE:	**APM2B10/APM02B2** INTRODUCTION TO NUMERICAL ANALYSIS
CAMPUS:	**AUCKLAND PARK KINGSWAY**

NOVEMBER EXAMINATION

DATE: 12/11/2018 **SESSION: 08:30 – 10:30**

ASSESSOR **DR MV VISAYA AND MR JM HOMANN**
MODERATOR **PROF F CINTI**

MARKS: 40

NUMBER OF PAGES	3 PAGES (INCLUDING COVER PAGE)
INSTRUCTIONS	ANSWER ALL QUESTIONS. SHOW ALL CALCULATIONS. POCKET CALCULATORS MAY BE USED. WORK TO AT LEAST THREE DECIMAL PLACES. SYMBOLS HAVE THEIR USUAL MEANING. ALL ANGLES MEASURED IN RADIANS.

Question 1 (6 marks)

a) Derive

$$y_i''' = \frac{y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}}{2h^3} + \mathcal{O}\left(h^2\right).$$

b) Make use of formulae (from the Information sheet) with the lowest order of error to complete the following table.

i	0	1	2	3	4	5
x_i	0	0.15	0.3	0.4	0.5	0.6
y_i	0	0.149	0.269	0.389	0.479	0.565
y_i'		0.989	0.703		0.877	

Question 2 (14 marks)

a) Using composite trapezium rule, determine the number of steps (n) and step size (h) required to determine

$$\int_0^2 e^{-x^2} dx$$

to an accuracy of 0.1.

b) Show that given $n+1$ equidistant points $x_0, x_1, \ldots, x_n$, the formula for composite trapezium is given by

$$\int_{x_0}^{x_n} f(x)dx \approx \frac{h}{2}(y_0 + y_n + 2\sum_{i=1}^{n-1} y_i)$$

c) Using your answer n in a), use composite trapezium to determine an approximation to $\int_0^2 e^{-x^2} dx$.

Question 3 (12 marks)

Consider the differential equation

$$\frac{dy}{dt} = t\left(2y+1\right)$$

with initial value $y\left(0\right) = 1$.

a) Using equation (*) in the Information, show that

$$F\left(t, y\right) = \frac{1}{4}\left(10yt + 5t + 4yt^2 + 2t^2 + 2y + 1\right).$$

b) Approximate a solution to the differential equation on the interval $[0, 1]$ using the RK2 method with step size $h = 0.5$.

c) Show that

$$y''' = 8yt + 4t + 4ty + 2t + 8yt^3 + 4t^3.$$

d) Determine the upper bound on the magnitude of the global error of the differential equation at $t = 1$, where the local error is given by

$$-\frac{h^3}{6}y'''.$$

Question 4 (8 marks)

Using initial approximation $\mathbf{x}^{(0)} = (4, -2)$, apply Newton's method twice to the following systems of equations

$$(x-2)^2 + y^2 = 9$$
$$(x-3)^2 + y = 1$$

Information

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} + \mathcal{O}\left(h^2\right), \qquad y_i' = \frac{-y_{i+2} + 8y_{i+1} - 8y_{i-1} + y_{i-2}}{12h} + \mathcal{O}\left(h^4\right).$$

$$y_i' = \frac{y_{i+1} - y_i}{h} + \mathcal{O}\left(h\right), \qquad y_i' = \frac{-y_{i+2} + 4y_{i+1} - 3y_i}{2h} + \mathcal{O}\left(h^2\right).$$

$$y_i' = \frac{-y_{i-1} + y_i}{h} + \mathcal{O}\left(h\right), \qquad y_i' = \frac{y_{i-2} - 4y_{i-1} + 3y_i}{2h} + \mathcal{O}\left(h^2\right)$$

$$|\Delta| \leq \left|\frac{h^2(b-a)M}{12}\right|, \text{ where } M = \max_{[a,b]}\left|f''(x)\right|$$

$$|\Delta| \leq \left|\frac{h^4(b-a)K}{180}\right|, \text{ where } K = \max_{[a,b]}\left|f^{(4)}(x)\right|$$

$$y_{m+1} = y_m + hf(x_m, y_m)$$

$$\begin{aligned} k_1 &= hf(x_m, y_m) \\ k_2 &= hf(x_m + h, y_m + k_1) \\ y_{m+1} &= y_m + \frac{1}{2}(k_1 + k_2) \\ &= y_m + hF(x_m, y_m) \\ F &= \frac{1}{2}[f(x_m, y_m) + f(x_m + h, y_m + hf(x_m, y_m))] \end{aligned} \qquad (*)$$

$$\alpha_m \approx 1 + hF_y(x_m, y_m)$$

$$\begin{aligned} y' = f(x,y) \;&\Rightarrow\; y'' = f_x + ff_y \\ &\Rightarrow\; y''' = f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_x f_y + ff_y^2 \end{aligned}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial x}f(x_n, y_n) & \frac{\partial}{\partial y}f(x_n, y_n) \\ \frac{\partial}{\partial x}g(x_n, y_n) & \frac{\partial}{\partial x}g(x_n, y_n) \end{bmatrix} \begin{bmatrix} x_{n+1} - x_n \\ y_{n+1} - y_n \end{bmatrix} = -\begin{bmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{bmatrix}$$

where $\mathbf{x}^{(n)} = (x_n, y_n)$

$\mathbf{r}^{(n+1)} = \mathbf{x}^{(n+1)} - \mathbf{x}^{(n)}$

END OF QUESTION PAPER