UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE
FAKULTEIT NATUURWETENSKAPPE

DEPARTMENT OF MATHEMATICS

MODULE **MAT0CA2 / MATECA2**
Engineering Sequences, Series and Vector Calculus

CAMPUS **APK**

EXAM **JULY SUPPLEMENTARY EXAM 2017**

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DURATION 2 HOURS

MARKS 50

SURNAME AND INITIALS ____________________

STUDENT NUMBER ____________________

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INSTRUCTIONS:
1. ANSWER ALL QUESTIONS ON THE PAPER IN PEN
2. CALCULATORS ARE ALLOWED
3. INDICATE **CLEARLY** ANY ADDITIONAL WORKING OUT

Question 1

[5]

Find all real numbers k for which the sequence

$$\left\{(-1)^n \frac{1 \cdot 3 \cdots (2n+1)}{(2n)!} \pi^{3n} k^n\right\}_{n=1}^{\infty}$$

converges.

Suggested solution: Let a_n denote the nth term of the given sequence. Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(2n+3)\cdot \pi^3 \cdot |k|}{(2n+2)(2n+1)} = 0$$

for all $k \in \mathbb{R}$. Thus, by the Ratio Test it follows that the series

$$\sum_{n=1}^{\infty}(-1)^n \cdot \frac{1\cdot 3\cdots(2n+1)}{(2n)!}\cdot \pi^{3n}\cdot k^n$$

converges for all $k \in \mathbb{R}$. Hence,

$$\lim_{n\to\infty}(-1)^n \cdot \frac{1\cdot 3\cdots(2n+1)}{(2n)!}\cdot \pi^{3n}\cdot k^n = 0$$

for all $k \in \mathbb{R}$. Therefore, the given sequence converges for all $k \in \mathbb{R}$.

Question 2

State the Integral Test for series. [4]

Suggested solution: Suppose f is a continuous, positive and decreasing function on $[1, \infty)$ and let $a_n = f(n)$ for all $n \in \mathbb{N}$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\, dx$ is convergent. In other words:

(i) If $\int_1^{\infty} f(x)\, dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)\, dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Question 3

[10]

By using an appropriate method, determine whether the following series converge or diverge:

(3.1) $\sum_{n=1}^{\infty} \frac{\arctan n}{n^{3/2}}$ (4)

Suggested solution: By the properties of the arctan function, it follows that the given series has positive terms. Hence, the Comparison Tests apply.

Conseider the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ and observe that it is convergent since $p = \frac{3}{2} > 1$. Moreover, notice that

$$\lim_{n\to\infty} \frac{\arctan n}{n^{3/2}} \cdot \frac{n^{3/2}}{1} = \lim_{n\to\infty} \arctan n = \frac{\pi}{2} > 0.$$

Hence, by the Limit Comparison Test it follows that the given series is convergent.

(3.2) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$ (3)

(3.3) $\sum_{n=1}^{\infty} \frac{(-5)^{2n}}{n^2\, 9^n}$ (3)

Question 4

[4]

Find the radius and interval of convergence of the series:

$$\sum_{n=0}^{\infty} \frac{2^n (x-3)^n}{\sqrt{n+3}}$$

Question 5

[3]

Find the sum of the series:

$$-\pi + \frac{\pi^2}{2} - \frac{\pi^3}{3!} + \frac{\pi^4}{4!} - \cdots$$

Suggested solution: We have

$$\begin{aligned}
-\pi + \frac{\pi^2}{2} - \frac{\pi^3}{3!} + \frac{\pi^4}{4!} - \cdots &= -1 + 1 - \pi + \frac{\pi^2}{2} - \frac{\pi^3}{3!} + \frac{\pi^4}{4!} - \cdots \\
&= -1 + \sum_{n=0}^{\infty} \frac{(-\pi)^n}{n!} \\
&= -1 + e^{-\pi},
\end{aligned}$$

where we have used the fact that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ for all } x \in \mathbb{R}.$$

Question 6

[4]

Use series to approximate

$$\int_0^1 \cos\left(x^4\right)\, dx$$

correct to three decimal places.

Question 7

[3]

If v is the speed of a particle along a curve C, $\mathbf{T}$ and $\mathbf{N}$ the unit tangent and unit normal vectors respectively of the particle's position vector $\mathbf{r}$, and κ is the curvature of C, then show that the acceleration $\mathbf{a}$ of the particle is given by

$$\mathbf{a} = v'\,\mathbf{T} + \kappa v^2\,\mathbf{N}$$

Question 8

Calculate the curvature of the curve $f(x) = e^x$ in the plane. [3]

Question 9

[5]

Find the velocity, acceleration and speed of a particle with the given position function. Sketch the path of the particle and draw the velocity and acceleration vectors for the specified value of t.

$$r(t) = 3\cos t\ \mathbf{i} + 2\sin t\ \mathbf{j}\ ; \quad t = \frac{\pi}{3}$$

Question 10

[5]

If it is given that

$$\mathbf{u}(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}''(t)]$$

then show

$$\mathbf{u}'(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}'''(t)]$$

for $\mathbf{r}$ and $\mathbf{u}$ arbitrary position vectors.