FACULTY OF SCIENCE
FAKULTEIT NATUURWETENSKAPPE

DEPARTMENT OF PURE AND APPLIED MATHEMATICS

MODULE	**MAT0CA2 / MATECA2** Engineering Sequences, Series and Vector Calculus
CAMPUS	**APK**
EXAM	**MAY 2017**

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DURATION	2 HOURS
MARKS	50

SURNAME AND INITIALS ______________________________

STUDENT NUMBER ______________________________

CONTACT NUMBER ______________________________

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INSTRUCTIONS:

1. ANSWER ALL QUESTIONS ON THE PAPER IN PEN
2. CALCULATORS ARE ALLOWED
3. INDICATE CLEARLY ANY ADDITIONAL WORKING OUT

Question 1

[8]

For questions (1.1) - (1.8), please circle only **ONE** correct answer:

(1.1) Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{3\pi}{3^n}$

(a) π (b) $\frac{\pi}{4}$ (c) $\frac{5\pi}{6}$ (d) $\frac{2\pi}{3}$ (e) $\frac{3\pi}{2}$

(1.2) If $\lim_{n\to\infty} |a_n| = 0$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(a) True (b) False

(1.3) Given the series

$$A := \sum_{k=1}^{\infty} \ln\left(\frac{3k^2+1}{2k^2+k}\right) \quad \text{and} \quad B := \sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)^n$$

(a) A is convergent, B is divergent.

(b) A is divergent, B is convergent.

(c) The series are both convergent.

(d) The series are both divergent.

(1.4) Suppose $\sum a_n$ is convergent and that $\sum b_n$ is divergent. Which of the following statements are always true?

(i) $\lim_{n\to\infty} a_n = 0$.

(ii) $\lim_{n\to\infty} b_n \neq 0$.

(iii) If $\sum(a_n + b_n)$ is divergent.

(iv) If $\sum(a_n - b_n)$ is convergent.

(a) i, ii & iii (b) ii & iv (c) i & iii (d) i & iv (e) ii & iii

(1.5) The Taylor series of e^x about the point $x = -1$ is given by:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{e^n(n!)}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(x+1)^n}{e(n!)}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(x+1)^n}{(e^2 n)!}$

(1.6) Let C be a smooth curve defined by a vector function $\mathbf{r}$ with unit tangent vector $\mathbf{T}$, binormal vector $\mathbf{B}$ and normal vector $\mathbf{N}$. Then the following statements are true:

(i) $\mathbf{T} \perp \mathbf{T}'$

(ii) $\mathbf{T} \perp \mathbf{B}$

(iii) $\mathbf{B} \perp \mathbf{N}$

(a) i & ii (b) ii & iii (c) i, ii & iii

(1.7) Let C be a smooth curve defined by a vector function $\mathbf{r}$. The curvature of the curve C is defined as:

(i) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$

(ii) $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$

(iii) $\kappa(x) = \dfrac{|f''(x)|}{[1-(f'(x))^2]^{3/2}}$

(a) i & ii (b) ii & iii (c) i, ii & iii

(1.8) If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are differentiable vector functions, then

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}'(t)$$

(a) True

(b) False

Question 2

[6]

Using an appropriate method, determine whether the following series are convergent or divergent. If the series converges, find its sum:

(2.1) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{\arctan(n+1)}{\arctan(n)}\right)$ (3)

(2.2) $\sum_{n=2}^{\infty} \frac{a \ln n}{(n+1)^2}$ (3)

Question 3 [5]

State and prove the Limit Comparison Test for series.

Question 4

[5]

If k is a positive integer, find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(kn)!}{(n!)^k}(x-k)^n$$

Question 5 [5]

Use a power series to approximate the definite integral correct to within 10^{-9}:

$$\int_0^{0.1} \left(\frac{x}{x+1}\right)^3 dx$$

Question 6

[5]

(6.1) Expand $f(x) = \dfrac{x}{(1-x)^2}$ as a power series. (3)

(6.2) Use (6.1) to find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$. (2)

Question 7 [4]

Consider the following vector-valued function

$$\mathbf{r}(t) = \frac{\cos t}{\sqrt{1+t^2}}\,\mathbf{i} + \frac{\sin t}{\sqrt{1+t^2}}\,\mathbf{j} + \frac{-t}{\sqrt{1+t^2}}\,\mathbf{k}$$

(7.1) Show that the curve C defined by $\mathbf{r}(t)$ lies on the sphere $x^2 + y^2 + z^2 = 1$. (1)

(7.2) Determine $\mathbf{r}'(0)$. (2)

(7.3) What can you conclude, if anything, about the angle between $\mathbf{r}(t)$ and $\mathbf{r}'(t)$? (1)

Question 8 [1]

Prove that the curvature of the curve given by the vector function $\mathbf{r}(t)$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Question 9

[3]

Reparametrise the following curve with respect to arc length measured from the point $t = 0$ in the direction of increasing t:

$$\mathbf{r}(t) = 3\cos t\ \mathbf{i} + 3\sin t\ \mathbf{j} + 5t\ \mathbf{k}$$

Question 10 [5]

Let $\mathbf{v}$ be the velocity, v the speed and $\mathbf{a}$ the acceleration of a particle whose position is given by the vector function $\mathbf{r}$.

(10.1) Prove that $\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}$. (3)

(10.2) Let C be the curve given by the position vector $\mathbf{r}(t) = \langle e^t, \sqrt{2}t, e^{-t}\rangle$. Show that the tangential component of the acceleration of a particle moving along the curve C is given by $a_T = e^t - e^{-t}$. (2)