UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE

DEPARTMENT OF MATHEMATICS

MAT0AB2

ENGINEERING LINEAR ALGEBRA B

SUPPLEMENTARY EXAM

EXAMINERS: Ms C. Le Roux, Dr J. Mba

MODERATOR: Dr G. Braatvedt

TIME: **120** MINUTES — **60** MARKS

SURNAME AND INITIALS: ____________________

STUDENT NUMBER: ____________________

TEL NO.: ____________________

INSTRUCTIONS:

1. The paper consists of **10** printed pages, **excluding** the front page.
2. Answer all questions.
3. **Write out all calculations (steps) and motivate all answers.**
4. Read the questions carefully.
5. Questions are to be answered on the question paper in the space provided. Please indicate when the blank side of a page is used.
6. **No calculators are allowed.**
7. Good luck!

Question 1 [12]

Answer the following **True and False** questions AND give a short justification/counterexample respectively:

a) The dimension of a vector space V is defined by the number of vectors in the space. (2)

TRUE	
FALSE	

b) Let $S = \{\overline{v_1}, \overline{v_2}\}$ be a subset of the vector space $\mathbb{R}^3$. Suppose that S is linearly independent, but not a basis for $\mathbb{R}^3$. Then we may add any vector $\overline{v} \notin span(S)$ to S to produce a basis for $\mathbb{R}^3$. (2)

TRUE	
FALSE	

c) Any basis for the vector space P_n will have n basis vectors. (2)

TRUE	
FALSE	

d) If a matrix A is orthogonally, then A is symmetric. (2)

TRUE	
FALSE	

e) If a matrix A is orthogonally diagonalizable, then 0 is an eigenvalue of A. (2)

TRUE	
FALSE	

f) If an $n \times n$ matrix is symmetric, then it has n distinct eigenvalues. (2)

TRUE	
FALSE	

Question 2 [4]

Let $W = \mathsf{span}((1, 0, -1, 2), (1, 2, 0, 0))$ be a subspace of $\mathbb{R}^4$ with the Euclidean inner product.

(a) Find a basis for $W^\perp$. (2)

(b) Does $(1, 1, 1, 1)$ belong to W, $W^\perp$, both or neither? **Motivate your answer.** (2)

Question 3 [3]

Consider the subspace $V = \{(x, y, z) \in \mathbb{R}^3 \mid y - 2z = 0\}$ of $\mathbb{R}^3$.

a) Find a basis for V. (2)

b) Find the dimension of V. (1)

Question 4 [4]

Let $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix}$.

a) Find the basis B if $P = P_{B \to S}$, where S is the standard basis for $\mathbb{R}^3$. (1)

b) Find the basis B if $P = P_{S \to B}$, where S is the standard basis for $\mathbb{R}^3$. (3)

Question 5 [7]

Consider the matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$

a) Determine whether A is diagonalizable. If A is diagonalizable, then find a matrix P that diagonalizes A. If A is not diagonalizable, explain why not. (4)

b) Find the geometric and algebraic multiplicity of each eigenvalue of A. (3)

Question 6 [6]

Consider the matrix transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$T(3,1) = (1,2) \text{ and } T(2,1) = (-1,1)$$

(a) Find $T(1,0)$ and $T(0,1)$ (2)

(b) Find the standard matrix $[T]$ (1)

(c) Show that T is one-to-one. (1)

(d) Determine the formula to calculate $T(x_1, x_2)$. (2)

Question 7 [6]

Consider the linear differential system:

$$\begin{aligned} y_1' &= 3y_1 - y_2 \\ y_2' &= -y_1 + 3y_2 \end{aligned}$$

a) Solve the above system using the diagonalization method. (4)

b) Find a solution that satisfies the initial conditions $y_1(0) = -1$ and $y_2(0) = 1$. (2)

Question 8 [5]

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ and $\bar{b} = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$

a) Find the QR-decomposition of A. (3)

b) Use the decomposition in a) to solve the least squares problem $A\bar{x} = \bar{b}$. (2)

Question 9 [6]

Let Q be the following quadratic form:

$$Q(x_1, x_2) = 3x_1^2 + 3x_2^2 - 2x_1x_2$$

a) Find a symmetric matrix A such that $\bar{x}^T A\bar{x}$ represents the quadratic form Q. (1)

b) Orthogonally diagonalize the matrix A found in a). (3)

c) Find an orthogonal change of variables that eliminates the cross product terms in the quadratic form Q, and express Q in terms of the new variables. (1)

d) Identify the conic section represented by the quadratic form obtained in c). (1)

Question 10 [7]

Let $A = \begin{bmatrix} 1 & 2-i \\ 2+i & -3 \end{bmatrix}$

a) Let $B = \mathrm{Im}(A)$. Find a matrix P and the matrix $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that $B = PCP^{-1}$. (3)

b) Find a unitary matrix P' that diagonalizes the matrix A. (4)