UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE

DEPARTMENT OF PURE AND APPLIED MATHEMATICS

MODULE:	ASMA1A1
COURSE:	CALCULUS OF ONE VARIABLE FUNCTIONS (ALTERNATIVE SEMESTER)
CAMPUS:	APK
EXAM:	SUPPLEMENTARY EXAM 2018

DATE: 12/01/2018

TIME: 08:00 – 10:00

ASSESSOR: MR SD NGIDI

INTERNAL MODERATOR: DR A CRAIG

DURATION: 2 HOURS

MARKS: 70

SURNAME AND INITIALS

MEMO

STUDENT NUMBER

CONTACT NUMBER

NUMBER OF PAGES: 1+12 PAGES (including front page)

INSTRUCTIONS: ANSWER ALL THE QUESTIONS IN PEN

NO CALCULATORS ALLOWED.

If you require extra space, continue on the adjacent blank page next to it and indicate this clearly.

Question 1 [10 marks]

For questions 1.1 - 1.10, choose **one** correct answer, and make a cross (X) in the correct block.

Question	a	b	c	d	e
1.1		X			
1.2				X	
1.3	X				
1.4			X		
1.5				X	
1.6		X			
1.7			X		
1.8			X		
1.9			X		
1.10				X	

1.1 Convert 240° to radians. [1]

a) $\frac{3\pi}{2}$

b) $\frac{4\pi}{3}$

c) $\frac{3\pi}{4}$

d) $\frac{1}{3\pi}$

e) None of the above

1.2 Convert the complex number $z = 1 - \sqrt{3}i$ into polar form. [1]

a) $\sqrt{2}(\cos\frac{4\pi}{3} - i\sin\frac{4\pi}{3})$

b) $\sqrt{3}(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3})$

c) $\sqrt{2}(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3})$

d) $2(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3})$

e) None of the above

1.3 The correct expansion of $\sum_{i=3}^{6}\frac{5^{i-2}}{i-2}$ is: [1]

a) $\frac{5}{1} + \frac{5^2}{2} + \frac{5^3}{3} + \frac{5^4}{4}$

b) $\frac{5^2}{2} + \frac{5^3}{3} + \frac{5^4}{4} + \frac{5^5}{5}$

c) $\frac{5^0}{1} + \frac{5^1}{1} + \frac{5^2}{2} + \frac{5^3}{3}$

d) $\frac{5^{-1}}{1} + \frac{5^0}{2} + \frac{5^1}{3} + \frac{5^2}{4}$

e) None of the above

1.4 If $f(x) = x^4 - 1$, $g(x) = \sqrt[4]{x^2 - 1}$ and $h(x) = \sqrt{x + 2}$, then $(f \circ g \circ h)(x)$ is. [1]

a) -2

b) 2

c) x

d) Does not exist

e) None of the above

1.5 $\lim_{x \to \infty} \arctan x =$ [1]

a) $-\frac{\pi}{4}$

b) $\frac{\pi}{4}$

c) $-\frac{\pi}{2}$

d) $\frac{\pi}{2}$

e) None of the above

1.6 $\lim_{x \to -\infty} e^x =$ [1]

a) 1

b) 0

c) ∞

d) $-\infty$

e) None of the above

1.7 If $f(x) = x^5 - 1$ then the inverse function $f^{-1}(x)$ is defined by: [1]

a) $\dfrac{1}{\sqrt[5]{x} + 1}$

b) $\dfrac{1}{\sqrt[5]{x + 1}}$

c) $\sqrt[5]{x + 1}$

d) $\sqrt[5]{x} + 1$

e) None of the above

1.8 Which one of the curves has a vertical asymptote? [1]

a) $y = \arctan x$

b) $y = \sqrt{x}$

c) $y = \ln x$

d) $y = e^x$

e) None of these

1.9 If f is differentiable, then $\frac{d}{dx}(f(\sqrt{x})) = ...$ [1]

a) $\frac{f'(x)}{2\sqrt{x}}$

b) $f'(x)\sqrt{x}$

c) $\frac{f'(\sqrt{x})}{2\sqrt{x}}$

d) $\frac{f'(x)}{\frac{1}{2}\sqrt{x}}$

e) None of the above

1.10 The negation of the conditional proposition $\neg p \rightarrow q$ is: [1]

a) $\neg p \rightarrow \neg q$

b) $p \wedge \neg q$

c) $\neg q \rightarrow p$

d) $\neg p \wedge \neg q$

e) None of the above

Question 2 [5 marks]

Use a truth table to show that $(p \to q) \to r$ and $p \to (q \to r)$ are not logically equivalent.

p	q	r	$p \to q$	$(p\to q)\to r$	$q \to r$	$p\to(q\to r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T
F	F	F	T	F	T	T

Not the same

Question 3 [5 marks]

a) State the Intermediate Value Theorem. [2]

Suppose that f is continuous on the close interval $[a,b]$, Let N be any number between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$.

Then there exists a number $c \in (a,b)$ such that $f(c) = N$.

b) Use the Intermediate Value Theorem to show that the function $f(x) = x^3 - x - 1$ has a root in the interval $(1, 2)$. [3]

The function is a polynomial, so its continuous on $\mathbb{R}$.

$\therefore$ Continuous on $(1,2)$

$f(1) = (1)^3 - 1 - 1 = -1$ $\qquad f(1) < 0 < f(2)$

$f(2) = (2)^3 - 2 - 1 = 5$

$\therefore$ There is a number c between 1 and 2 such that $f(c) = 0$.

$\therefore$ There is a root between 1 and 2.

Question 4 [6 marks]

a) Solve for x if: $\dfrac{x-1}{x+2} > -1$ [2]

$$\frac{x-1}{x+2} > -1$$

$$\frac{2x+1}{x+2} > 0$$

CP: $2x+1=0$

$x=-\frac{1}{2}$ ✓✓

$x+2=0$

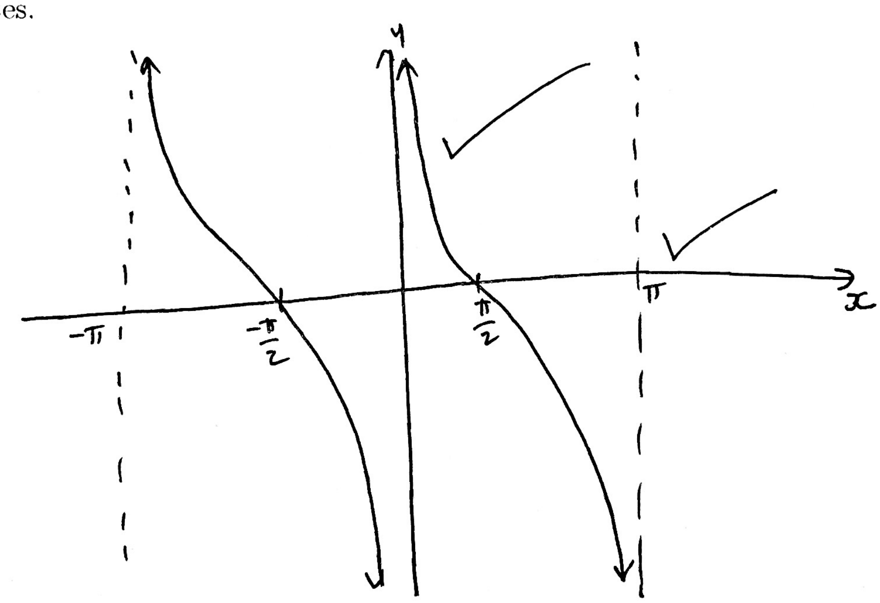
$x=-2$ ✓✓

$+$ ✓✓ $-$ $+$

-2 $\frac{1}{2}$

$x \in (-\infty,-2) \cup (-\frac{1}{2},\infty)$ ✓✓

b) Sketch the graph of $\cot x$ for $x \in [-\pi, \pi]$. Label all intercepts with axes as well as any asymptotes. [2]



c) Let $z = 2+5i$ and $w = 2-2i$. Find $\frac{z}{w}$ and write your answer in the form $a+bi$. [2]

$$\frac{2+5i}{2-2i} \times \frac{2+2i}{2+2i} ✓✓ = \frac{4+14i-10}{4+4} ✓✓$$

$$= \frac{4+14i+10i^2}{4-4i^2} = \frac{-6+14i}{8} = -\frac{6}{8}+\frac{14}{8}i ✓$$

Question 5 [4 marks]

Determine (without using L'Hospital's Rule):

a) $\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h}$ [2]

$$= \lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} \times \frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}$$

$$= \lim_{h\to 0} \frac{1+h-1}{h(\sqrt{1+h}+1)}$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{1+h}+1)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{1+h}+1}$$

$$= \frac{1}{\sqrt{1+0}+1}$$

$$= \frac{1}{2}$$

b) $\lim_{x\to -3} \frac{x+3}{x^3+27}$ [2]

$$= \lim_{x\to -3} \frac{x+3}{(x+3)(x^2-3x+9)}$$

$$= \lim_{x\to -3} \frac{1}{x^2-3x+9}$$

$$= \frac{1}{27}$$

Question 6 [6 marks]

Evaluate the following limits (use L'Hospital's Rule when needed):

a) $\lim_{x\to\infty} \frac{\ln x}{x} \to \frac{\infty}{\infty}$ [2]

$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$

$= 0$

b) $\lim_{x\to 0^+} [\cos(2x)]^{\frac{1}{x^2}}$ [4]

$y = [\cos(2x)]^{\frac{1}{x^2}}$

$\ln y = \ln [\cos(2x)]^{\frac{1}{x^2}}$

$\ln y = \frac{1}{x^2} \cdot \ln[\cos(2x)] = \frac{\ln(\cos(2x))}{x^2}$

$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \left(\frac{\ln[\cos(2x)]}{x^2} \right) \to \frac{\ln(1)}{0} = \frac{0}{0}$

$\overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{-2\sin(2x)}{\cos(2x)}}{2x} = \lim_{x\to 0^+} \frac{-\tan(2x)}{x} \to \frac{0}{0}$

$\overset{L'H}{=} \lim_{x\to 0^+} \frac{-2\sec^2(2x)}{1} = -2$

$\therefore y = e^{\ln x}$

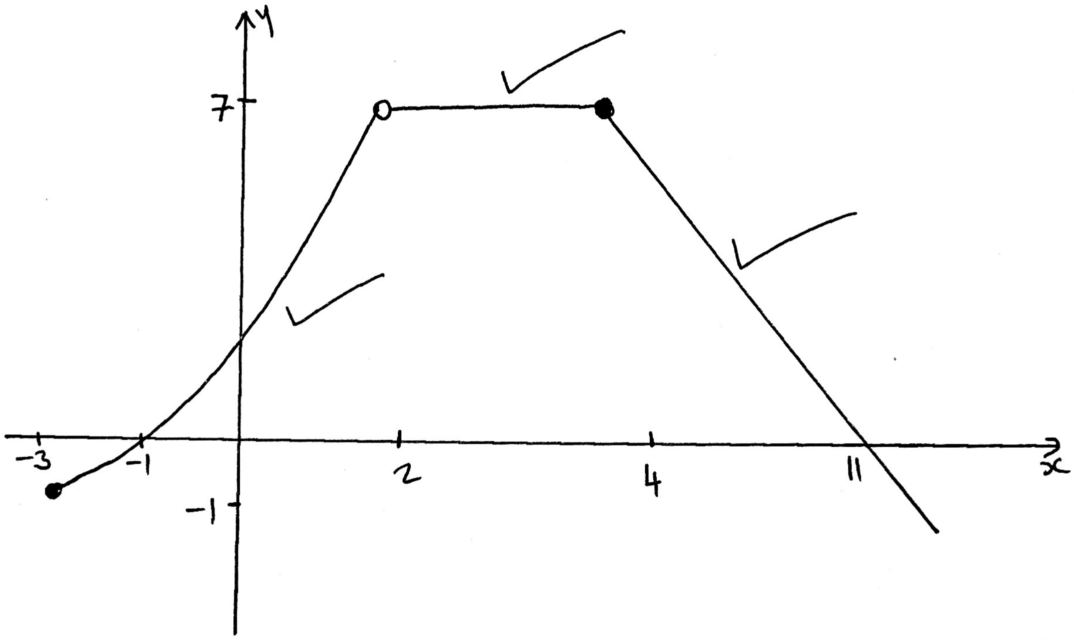
$\therefore y = e^{-2}$

Question 7 [9 marks]

Given the following case-defined function:

$$f(x) = \begin{cases} 2^{x+1} - 1 & \text{if } -3 \leq x < 2 \\ 7 & \text{if } 2 < x < 4 \\ -x + 11 & \text{if } x \geq 4 \end{cases}$$

a) Sketch the graph of $f(x)$. [3]



Use the graph to answer the following limit questions:

b) Determine $\lim_{x \to 2^-} f(x)$. [1]

7

c) Determine $\lim_{x \to 2^+} f(x)$. [1]

7

d) Determine $\lim_{x \to 2} f(x)$. [1]

7

e) Determine $f(2)$. [1]

Not defined

f) Is $f(x)$ continuous at $x = 2$? Motivate your answer. [2]

No, not defined at 2.

Question 8 [8 marks]

Differentiate the following functions:

a) $y = 7xe^{x^2}$ [2]

$y' = 7\left((1)e^{x^2} + xe^{x^2} \cdot 2x\right)$

$= 7e^{x^2} + 2x^2e^{x^2}$

b) $y = \sin(x+y) + \cos x$ [3]

$\frac{d}{dx}(y) = \frac{d}{dx}\left(\sin(x+y) + \cos x\right)$

$y' = \cos(x+y) \cdot (1+y') + (-\sin x)$

$y' = \cos(x+y) + y'\cos(x+y) - \sin x$

$y' - y'\cos(x+y) = \cos(x+y) - \sin x$

$\therefore \quad y' = \dfrac{\cos(x+y) - \sin x}{1 - \cos(x+y)}$

c) $y = \dfrac{2x - e^{\sqrt{x}}}{1 + \tan x}$, you do not need to simplify the answer. [3]

$y = \dfrac{2x - e^{x^{1/2}}}{1 + \tan x}$

$y' = \dfrac{\left(2 - e^{x^{1/2}} \cdot \left(\frac{1}{2}x^{-\frac{1}{2}}\right)\right)(1+\tan x) - \left(2x - e^{x^{1/2}}\right)(\sec^2 x)}{(1+\tan x)^2}$

Question 9 [5 marks]

a) Prove the following Theorem: "If f is differentiable at a, then f is continuous at a". [4]

To prove : f is continuous

$$\therefore \lim_{x \to a} f(x) = f(a)$$

We show that $f(x) - f(a)$ approaches 0.

Rewrite $f(x) - f(a)$ as

$$f(x) - f(a) = \frac{f(x) - f(a)}{x-a} \times x-a$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x-a} \times \lim_{x \to a} (x-a)$$

$$= f'(a) \times 0$$

$$= 0$$

Now we go back to $\lim_{x \to a} f(x) = \lim_{x \to a} \left(f(x) - f(a) + f(a) \right)$

$$\lim_{x \to a} f(x) = \lim_{x \to a} \left(f(x) - f(a) \right) + \lim_{x \to a} f(a)$$

$$= 0 + f(a)$$

$$= f(a)$$

$$\therefore \lim_{x \to a} f(x) = f(a)$$

$\therefore f(x)$ is continuous

b) Is the converse of the Theorem in 9(a) true? [1]

No

Question 10 [3 marks]

a) Use proof by contrapositive to prove the following:
"Let $n \in \mathbb{Z}$, if $n^2 + 5$ is odd, then n is even." [3]

n is not even

n is odd

$n = 2k+1$ for some integer k

$n^2 + 5 = (2k+1)^2 + 5 = 2(k^2 + 2k + 3)$

$= 2M \quad M \in \mathbb{Z}$

$n^2 + 5$ is even

Question 11 [2 marks]

State the Fundamental Theorem of Calculus Part 1. [2]

If f is continuous on $[a,b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt, \quad a \le x \le b$$

is continuous on $[a,b]$ and differentiable on (a,b),

and

$$g'(x) = f(x)$$

Question 12 [7 marks]

Evaluate the following integrals

a) $\int -3x^2 + x - 5 \, dt$ [2]

$= \frac{-3x^3}{3} + \frac{x^2}{2} - 5x + C$

$= -x^3 + \frac{1}{2}x^2 - 5x + C$

b) $\int_0^{\frac{\pi}{4}} \sec\theta \tan\theta \, d\theta$ [2]

$= \sec\theta \Big]_0^{\pi/4}$

$= \sqrt{2} - 1$

c) $\int (2x + 2)e^{2x^2+4x} \, dx$ [3]

$= \int (2x+2)e^u \cdot \frac{du}{2(2x+2)}$

Let $u = 2x^2 + 4x$

$\frac{du}{dx} = 4x + 4$

$dx = \frac{du}{4x+4} = \frac{du}{2(2x+2)}$.

$= \frac{1}{2}\int e^u \cdot du$

$= \frac{1}{2}e^u + C$

$= \frac{1}{2}e^{2x^2+4x} + C$