UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE

DEPARTMENT:	PURE AND APPLIED MATHEMATICS
MODULE:	APM2B10/APM02B2 INTRODUCTION TO NUMERICAL ANALYSIS
CAMPUS:	AUCKLAND PARK KINGSWAY

NOVEMBER EXAMINATION

DATE: 13/11/2017	SESSION: 08:30 – 11:30
ASSESSOR	DR MV VISAYA and MR JM HOMANN
MODERATOR	DR JSC PRENTICE
DURATION: 2 HOURS	MARKS: 50

NUMBER OF PAGES	1 COVER PAGE 1 QUESTION PAGE 1 INFORMATION SHEET
INSTRUCTIONS	ANSWER ALL THE QUESTIONS. SHOW ALL CALCULATIONS. POCKET CALCULATORS MAY BE USED. WORK TO A PRECISION OF AT LEAST THREE DECIMAL PLACES. SYMBOLS HAVE THEIR USUAL MEANING. ALL ANGLES ARE MEASURED IN RADIANS.

QUESTION 1

Given $f(x) = \sin x - x$ and $x_0 = 0.6$, approximate a non-zero root of $f(x)$ by using Newton's Method subject to an error tolerance of $\epsilon = 10^{-3}$. **[5]**

QUESTION 2

Expand the step function

$$f(x) = \begin{cases} 5 & \text{when} \quad x \geq 0 \\ 0 & \text{when} \quad x < 0 \end{cases}$$

in terms of Chebyshev polynomials. **[8]**

QUESTION 3

Derive

$$y_i''' = \frac{-y_{i-3} + 3y_{i-2} - 3y_{i-1} + y_i}{h^3}$$

and determine the order of the error. **[7]**

QUESTION 4

Consider the integral

$$I = \int_{-1}^{1} e^{-x^2}\,\mathrm{d}x.$$

a) Determine an upper bound for the error in I in the case of applying the Composite Trapezium Rule.

b) Using (a), determine the step size and the minimum number of intervals required so that the error is less than 0.2.

c) Use (b) and apply the Composite Trapezium Rule to approximate I.

[15]

QUESTION 5

a) Given the differential equation

$$\frac{dy}{dx} = x - y$$

with initial value $x(1) = 1$, use the Runge-Kutta method of order two (RK2) to find A and B in the following table.

i	0	1	2	3	4
x_i	1	1.5	2	2.5	3
y_i	1	A	B	1.74414	2.15259

b) Using information in a), show that the characteristic function of RK2 is given by

$$F(x, y) = \frac{1 + 3x - 3y}{4}.$$

c) Assuming that the local error in RK2 has the form

$$\epsilon_{i+1} = \frac{h^3}{6} y'''(\xi), \quad \xi \in (x_i, x_{i+1}),$$

estimate an upper bound for the global error at $x = 3$. **[15]**

Information

$$f(x) = \frac{c_0}{2}T_0(x) + c_1T_1(x) + c_2T_2(x) + \cdots$$

$$c_k = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}}\, \mathrm{d}x$$

$$c_k = \frac{2}{\pi}\int_{0}^{\pi} f(\cos\theta)\cos(k\theta)\, \mathrm{d}\theta.$$

$$\int_a^b f(x)\mathrm{d}x \approx \frac{h}{2}\left(y_0 + y_N + 2\sum_{i=1}^{N-1} y_i\right)$$

$$|\Delta| \le \left|\frac{h^2(b-a)M}{12}\right|, \text{where} \quad M = \max_{[a,b]}\left|f''(x)\right|$$

$$\int_a^b f(x)\mathrm{d}x \approx \frac{h}{3}\left(y_0 + y_{2N} + 4y_1 + \sum_{i=1}^{N-1}(2y_{2i} + 4y_{2i+1})\right)$$

$$|\Delta| \le \left|\frac{h^4(b-a)K}{180}\right|, \quad K = \max_{[a,b]}\left|f^{(4)}(x)\right|$$

$$y_{m+1} = y_m + hf(x_m, y_m)$$

$$\begin{aligned} k_1 &= hf(x_m, y_m) \\ k_2 &= hf(x_m + h, y_m + k_1) \\ y_{m+1} &= y_m + \frac{1}{2}(k_1 + k_2) \\ &= y_m + hF(x_m, y_m) \\ F &= \frac{1}{2}\left[f(x_m, y_m) + f(x_m + h, y_m + hf(x_m, y_m))\right] \end{aligned}$$

$$\alpha_m \approx 1 + hF_y(x_m, y_m)$$

$$\begin{aligned} y' = f(x,y) \quad &\Rightarrow \quad y'' = f_x + ff_y \\ &\Rightarrow \quad y''' = f_{xx} + 2ff_{xy} + f^2f_{yy} + f_xf_y + ff_y^2 \end{aligned}$$