UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE

DEPARTMENT:	**PURE AND APPLIED MATHEMATICS**
MODULE:	**APM2B10/APM02B2** INTRODUCTION TO NUMERICAL ANALYSIS
CAMPUS:	**AUCKLAND PARK KINGSWAY**

SSA EXAMINATION

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MODERATOR **DR JSC PRENTICE**

DURATION: 2 HOURS **MARKS: 50**

NUMBER OF PAGES	1 COVER PAGE 1 QUESTION PAGE 1 INFORMATION SHEET
INSTRUCTIONS	ANSWER ALL THE QUESTIONS. SHOW ALL CALCULATIONS. POCKET CALCULATORS MAY BE USED. WORK TO A PRECISION OF AT LEAST THREE DECIMAL PLACES. SYMBOLS HAVE THEIR USUAL MEANING. ALL ANGLES ARE MEASURED IN RADIANS.

QUESTION 1

Given $f(x) = \sin x - x$ and $x_0 = 0.6$, approximate a non-zero root of $f(x)$ by using Newton's Method subject to an error tolerance of $\epsilon = 10^{-3}$. **[5]**

QUESTION 2

Consider the expansion of the step function

$$f(x) = \begin{cases} 0 & \text{when} \quad x \geq 0 \\ 5 & \text{when} \quad x < 0 \end{cases}$$

in terms of Chebyshev polynomials. Determine the form of the coefficients c_k. **[8]**

QUESTION 3

Derive

$$y_i'' = \frac{-y_{i-3} + 4y_{i-2} - 5y_{i-1} + 2y_i}{h^2} + O(h^2).$$

[7]

QUESTION 4

Consider the integral

$$I = \int_0^{\frac{\pi}{2}} e^{\sin x}\,\mathrm{d}x.$$

a) Determine an upper bound for the error in I in the case of applying the Composite Simpson's Rule (HINT: $f^{(3)}(x) = e^{\sin x}\left(\cos^3 x - 3\cos x \sin x - \cos x\right)$ and $f^{(4)}(x)$ has one critical point on $\left[0, \frac{\pi}{2}\right]$, namely $x = 0.4716$).

b) Using (a), determine the step size and determine the minimum number of intervals required so that the error is less than 10^{-3}.

c) Use (b) and apply the Composite Simpson's Rule to approximate I.

[15]

QUESTION 5

Apply Euler's method to the differential equation

$$\frac{dy}{dx} = \frac{x}{y}$$

with initial value $y(0) = 1$, find A, B, and C in the following table

i	0	1	2	3	4	5
x_i	0	0.1	0.2	0.3	0.4	0.5
y_i	1	A	B	1.0298	1.0589	C

b) Assuming that the local error has the form

$$\epsilon_{i+1} = -\frac{h^2}{2}y''(\xi), \quad \xi \in (x_i, x_{i+1})$$

estimate an upper bound for the global error at $x = 0.5$. **[15]**

Information

$$f(x) = \frac{c_0}{2}T_0(x) + c_1T_1(x) + c_2T_2(x) + \cdots$$

$$c_k = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}}\,\mathrm{d}x$$

$$c_k = \frac{2}{\pi}\int_{0}^{\pi} f(\cos\theta)\cos(k\theta)\,\mathrm{d}\theta.$$

$$\int_a^b f(x)\mathrm{d}x \approx \frac{h}{2}\left(y_0 + y_N + 2\sum_{i=1}^{N-1} y_i\right)$$

$$|\Delta| \le \left|\frac{h^2(b-a)M}{12}\right|, \text{where} \quad M = \max_{[a,b]}\left|f''(x)\right|$$

$$\int_a^b f(x)\mathrm{d}x \approx \frac{h}{3}\left(y_0 + y_{2N} + 4y_1 + \sum_{i=1}^{N-1}(2y_{2i} + 4y_{2i+1})\right)$$

$$|\Delta| \le \left|\frac{h^4(b-a)K}{180}\right|, \quad K = \max_{[a,b]}\left|f^{(4)}(x)\right|$$

$$y_{m+1} = y_m + hf(x_m, y_m)$$

$$\begin{aligned}
k_1 &= hf(x_m, y_m)\\
k_2 &= hf(x_m + h, y_m + k_1)\\
y_{m+1} &= y_m + \frac{1}{2}(k_1 + k_2)\\
&= y_m + hF(x_m, y_m)\\
F &= \frac{1}{2}\left[f(x_m, y_m) + f(x_m + h, y_m + hf(x_m, y_m))\right]
\end{aligned}$$

$$\alpha_m \approx 1 + hF_y(x_m, y_m)$$

$$\begin{aligned}
y' = f(x,y) \quad &\Rightarrow \quad y'' = f_x + ff_y\\
&\Rightarrow \quad y''' = f_{xx} + 2ff_{xy} + f^2f_{yy} + f_xf_y + ff_y^2
\end{aligned}$$

END OF QUESTION PAPER