APPLIED MATHEMATICS

Introduction to Dynamics
APM01B1/APM1B10

November Examination: 23/11/2017

Duration: 2 hours **Marks:** 50
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Instructions:

1. Symbols have their usual meaning.
2. Physical quantities are in SI units and **angles are in radians.**
3. All calculations must be shown.
4. Pocket calculators are permitted.
5. Work to a precision of at least three decimal places.

Question 1 (12 marks)

(a) Show that for uniform circular motion

$$r = \rho$$

where r is the radius of the motion, ρ is the radius of curvature in the $(\hat{\tau}, \hat{n})$ coordinate system.

(b) The equation of the trajectory of the particle is given by

$$\boldsymbol{r}(t) = (-2 + 2\sin t)\,\hat{x} + (-1 + 2\cos(2t))\,\hat{y} \equiv (-2 + 2\sin t, -1 + 2\cos(2t))$$

For $t = \frac{1}{2}$, find

i) $\boldsymbol{v}$ and $\boldsymbol{a}$

ii) $\hat{\tau}$ and $\hat{n}$.

iii) the tangential and normal components of $\boldsymbol{a}$

iv) the radius of curvature and the centre of curvature $\boldsymbol{C} = \boldsymbol{r} + \rho\hat{n}$.

Solution:

(a) First note that, for uniform circular motion,

$$\hat{r} = -\hat{n}, \qquad \hat{\theta} = \hat{\tau}$$

Then from the velocity vector,

$$r\dot{\theta}\hat{\theta} = r\dot{\theta}\hat{\tau} = v\hat{\tau}, \qquad \Rightarrow \qquad v = r\dot{\theta}.$$

Then, matching the components of the acceleration, we find

$$\begin{aligned} +r\dot{\theta}^2\hat{n} &= \frac{v^2}{\rho}\hat{n} \\ r\dot{\theta}^2 &= \frac{r^2\dot{\theta}^2}{\rho} \\ \rho &= r. \end{aligned}$$

(b) i)

$$\begin{aligned} \boldsymbol{v} &= (1.75517, -3.36588) \\ \boldsymbol{a} &= (-0.958851, -4.32242) \end{aligned}$$

ii)

$$\begin{aligned} \hat{\tau} &= (0.46237, -0.886687) \\ \hat{n} &= (0.886687, 0.46237)\,. \end{aligned}$$

iii)

$$a_\tau = 3.38929, \qquad a_n = -2.84876$$

iv)

$$\rho = -5.05827, \qquad \boldsymbol{C} = (-5.52625, -2.25819)\,.$$

Question 2 (7 marks)

A particle of mass 2kg experiences a time-dependent force $\boldsymbol{F} = (3t^2, 2t - 1)$ for a time period $T > 0$. The initial velocity of the particle is $\boldsymbol{v}_i = (1, 0)$. If the final velocity $\boldsymbol{v}_f$ is horizontal, determine the final speed of the particle. Assume $t_i = 0$.

Solution: Impulse equation

$$\boldsymbol{I} = \left(T^3, T^2 - T\right) = 2\,(v_f, 0) - 2\,(1, 0)$$

One then finds the two equations

$$\begin{aligned} T^3 &= 2v_f - 2 \\ T^2 - T &= 0. \end{aligned}$$

Solving, we find $T = 0$, or $T = 1$. For $T = 1$, then $v_f = 3/2$.

Question 3 (11 marks)

A perturbed ideal gas is described by the equation of state

$$\psi\,(x, y, z) = \alpha \frac{x}{y} + 3z^2,$$

where α is a constant. The variables x, y, z and ψ represent the temperature, volume, mass and pressure of the gas respectively.

Calculate the directional derivative $\frac{\partial \psi}{\partial s}$ in the direction of $\hat{x} + \hat{y} + \hat{z}$ at the point $P = (1, 1, 1)$.

(a) Use the parameterization $\boldsymbol{r} = \boldsymbol{r}_0 + s\hat{s}$

(b) Use the formula $\nabla\psi \cdot \hat{s}$.

Solution:

(a) Calculate the unit vector

$$\hat{s} = \frac{1}{\sqrt{3}}\,(\hat{x} + \hat{y} + \hat{z})$$

Then the straight line is parameterized by

$$\boldsymbol{r} = \boldsymbol{r}_0 + s\hat{s}$$

where we choose $\boldsymbol{r}_0 = \overline{OP} = (1, 1, 1)$. This gives

$$\begin{aligned} x\,(s) &= 1 + \frac{s}{\sqrt{3}} \\ y\,(s) &= 1 + \frac{s}{\sqrt{3}} \\ z\,(s) &= 1 + \frac{s}{\sqrt{3}} \end{aligned}$$

Substituting, we find

$$\psi(s) = \alpha + 3\left(1 + \frac{s}{\sqrt{3}}\right)^2.$$

Then

$$\left.\frac{\partial \psi}{\partial s}\right|_{s=0} = \left.\frac{\partial \psi}{\partial s}\right|_{P=(1,1,1)} = \frac{6}{\sqrt{3}}.$$

(b) Calculate $\nabla\psi \cdot \hat{s}$.

$$\begin{aligned}
\nabla\psi &= \left(\alpha\frac{1}{y}, -\alpha\frac{x}{y^2}, 6z\right) \\
\nabla\psi \cdot \hat{s} &= \frac{1}{\sqrt{3}}\left(\alpha\frac{1}{y} - \alpha\frac{x}{y^2} + 6z\right) \\
\left.\nabla\psi \cdot \hat{s}\right|_{P=(1,1,1)} &= \frac{6}{\sqrt{3}}.
\end{aligned}$$

Question 4 (12 marks)

Calculate the $\int_\Gamma \boldsymbol{F} \cdot d\boldsymbol{r}$, with $\boldsymbol{F} = (y + 2z, x - 2y, x^2 y)$, and where the path Γ is defined by

$$\begin{aligned}
y &= x^2 + 1 \\
z &= 4x + y.
\end{aligned}$$

In the integral, the lower limit is $(0, 1, 1)$, and the upper limit is $(1, 2, 6)$.

Solution: The parametric equations for the path Γ are

$$\begin{aligned}
x(s) &= s \\
y(s) &= s^2 + 1 \\
z(s) &= s^2 + 4s + 1.
\end{aligned}$$

The lower bound $(0, 1, 1)$ corresponds to $s = 0$, the upper bound $(1, 2, 6)$ corresponds to $s = 1$. The integral becomes

$$\int_\Gamma \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 \boldsymbol{F}(s) \cdot \frac{d\boldsymbol{r}}{ds}\, ds.$$

where

$$\begin{aligned}
\boldsymbol{r} &= \left(s, s^2 + 1, s^2 + 4s + 1\right) \\
\frac{d\boldsymbol{r}}{ds} &= \left(1, 2s^2, 2s + 4\right) \\
\boldsymbol{F}(s) &= \left(3s^2 + 8s + 3, -s^2 + s - 2, s^4 + s^2\right).
\end{aligned}$$

Putting everything together

$$\int_\Gamma \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 \left[2s^5 + 4s^4 - 2s^3 + 9s^2 + 4s + 3\right]\, ds = \frac{259}{30}.$$

Question 5 (8 marks)

(a) Does the Principle of Work and Energy hold for all forces, or only for conservative forces?

(b) The coefficient of friction between the tyres of a braking car and the road is $\mu = 0.5$. The car travels down a plane with an incline of $\theta = 15^o$ with respect to the horizontal. Use the Principle of Work and Energy to calculate the distance travelled by the car after it comes to a complete stop. Assume the car has an initial speed of 10 m/s.

Solution:

(a) Applies to all forces

(b) Orientate the axes in your reference frame such that the x-axis is parallel to the incline, and that + x direction is in the direction of the car's motion. Then

$$d\boldsymbol{r} = (dx, 0, 0) = dx\hat{x}.$$

We only need the forces in the x-direction. The friction force is directed towards the $-\ x$ direction, while the x-component of the gravitational force is pointing in the $+\ x$ direction:

$$F_x = -mg\cos(\theta)\,\mu + mg\sin(\theta)\,.$$

Calculating the work done

$$\begin{aligned} W = \int \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_0^d (-mg\cos(\theta)\,\mu + mg\sin(\theta))\,dx \\ &= (-mg\cos(\theta)\,\mu + mg\sin(\theta))\,d \\ &= mgd\,(-\cos(\theta)\,\mu + \sin(\theta))\,. \end{aligned}$$

which is equal to the change in kinetic energy

$$\Delta T = 0 - \frac{1}{2}mv_i^2 = -\frac{1}{2}mv_i^2.$$

Using the work energy principle:

$$\begin{aligned} mgd\,(-\cos(\theta)\,\mu + \sin(\theta)) &= -\frac{1}{2}mv_i^2 \\ d &= -\frac{v_i^2}{2g\,(-\cos(\theta)\,\mu + \sin(\theta))} \\ &= 22.762 \text{ m}. \end{aligned}$$

Information

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\begin{aligned}
\boldsymbol{r} &= (x, y) = (r\cos\theta, r\sin\theta)\,. \\
\boldsymbol{v} &= v_r\hat{r} + v_\theta\hat{\theta} = v_\tau\hat{\tau} + v_n\hat{n} \\
&= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} = v\hat{\tau}. \\
\boldsymbol{a} &= a_r\hat{r} + a_\theta\hat{\theta} = a_\tau\hat{\tau} + a_n\hat{n} \\
&= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta} = \dot{v}\hat{\tau} + \left(\frac{v^2}{\rho}\right)\hat{n}. \\
v &= |\boldsymbol{v}|\,. \\
\hat{r} &= (\cos\theta, \sin\theta)\,, \qquad \hat{\theta} = (-\sin\theta, \cos\theta) \\
\hat{\tau} &= (\cos\psi, \sin\psi)\,, \qquad \hat{n} = (-\sin\psi, \cos\psi)
\end{aligned}
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\begin{aligned}
\boldsymbol{I} &= \int_{t_i}^{t_f} \boldsymbol{F}dt = m\boldsymbol{v}_f - m\boldsymbol{v}_i \\
d\boldsymbol{r} &= dx\hat{x} + dy\hat{y} + dz\hat{z} \\
\nabla &= \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \\
W &= \int_\Gamma \boldsymbol{F}\cdot d\boldsymbol{r} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2
\end{aligned}
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