Question 1 [14 marks]

Waves on a thin, flexible string of mass per unit length ρ and subject to a tension T are governed by the linear wave equation:

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2},$$

where $y(x,t)$ is the wave displacement at time t and position x, and $\frac{T}{\rho} = v^2$.

(1.1) Show that travelling waves of the form

$$y(x,t) = y(u),$$

where $u = x - vt$, may be solutions of the linear wave equation above, and find the two values of v that make them so. *[6 marks]*

(1.2) Is the wave function $y(x,t) = 4\ln(5x - 7t)$ a solution of the linear wave equation above? Prove your answer mathematically, and find the possible value(s) for the speed of propagation of the wave (v) from your proof. *[4 marks]*

(1.3) Consider the wave function $y(x,t) = x^2 + v^2t^2$.
(a) Show that $y(x,t)$ is a solution of the linear wave equation above. *[2 marks]*
(b) Show that $y(x,t)$ can be written as $y(x,t) = f(x+vt) + g(x-vt)$ and determine the functional forms of f and g. *[2 marks]*

Question 2 [16 marks]

(2.1) A sinusoidal wave propagating on a string is described by the wave function

$$y(x,t) = 0.150\,\sin\left(0.80\,x - 50.0\,t\right)$$

where x and y are in meters and t in seconds. The mass per unit length of the string is 12.0 g/m.
(a) Calculate the maximum transverse acceleration of an element of this string. *[3 marks]*
(b) Calculate the maximum transverse force on a segment of the string of length 1.0 cm. *[2 marks]*
(c) Calculate the tension in the string. *[2 marks]*

(2.2) Explain what it means mathematically for a wave equation to be defined as *linear*. *[2 marks]*

(2.3) Consider the following wave equation:

$$\frac{\partial y}{\partial t} + 6y\frac{\partial y}{\partial x} + \frac{\partial^3 y}{\partial t^3} = 0\,,$$

where the wave displacement at time t and position x is given by $y(x,t)$. Is this wave equation linear or not? Substantiate your answer mathematically and carefully explain your conclusion. *[7 marks]*

Question 3 [24 marks]

(3.1) Explain in your own words why complex waveforms are important, and why they are used as solutions of wave equations originating from real physical problems. *[2 marks]*

(3.2) The equation below

$$m\frac{d^2x}{dt^2} + kx + bv = 0$$

describes an harmonic oscillator moving in a resistive environment. Here m is the mass of the oscillator, k is the spring constant of the spring and b indicates the damping due to friction. Show that a complex exponential of the form

$$z = Ae^{i(pt+\alpha)}$$

(where A, α and p are parameters) to solve the equation above. In particular, show that this is the case for $p = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} + i\frac{\gamma}{2}$, so that the final solution has the form

$$z(t) = A\,e^{(-\gamma/2)t}e^{i(\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}t+\alpha)}.$$

(You should know what the parameters ω_0^2 and γ are.) *[9 marks]*

(3.3) Explain what the physical meaning of the real term $e^{(-\gamma/2)t}$ (in the proposed solution $z(t)$ above) is in terms of the amplitude of the oscillations. *[2 marks]*

(3.4) Explain how the solution proposed in question 3.2 changes in the case when $\omega_0^2 < \dfrac{\gamma^2}{4}$. Do we still have an oscillatory motion in this case? Explain your answer. *[5 marks]*

(3.5) The equation that represents forced oscillations is the following:

$$F_0 \sin \omega t - b\frac{dx}{dt} - kx = m\frac{d^2x}{dt^2}.$$

Show that the steady state solution has the form $x(t) = A\cos(\omega t + \phi)$, and find an expression for the amplitude of the oscillation A as a function of ω and ω_0 (or, in other words, the resonance formula). *[6 marks]* (*Please note*: $\omega_0^2 = k/m$.)

Question 4 [17 marks]

(4.1) The wave equation for shallow water gravity waves is

$$\frac{\partial^2 h}{\partial t^2} = gh_0\frac{\partial^2 h}{\partial x^2}$$

where g is the acceleration of gravity, h_0 is the water depth and $h(x,t)$ is the water displacement at time t and position x. Show that this wave equation allows for soliton-type solutions of the form

$$h(x,t) = 2\alpha^2\,\mathrm{sech}^2[\alpha(x - vt)]\,,$$

(where α is a constant parameter, and v is the propagation speed of the wave), and derive the expression for v as a function of g and h_0. *[9 marks]*
Please note: $\mathrm{sech}x = \dfrac{1}{\mathrm{cosh}x}$; $\dfrac{\mathrm{d}}{\mathrm{d}x}\mathrm{cosh}x = \mathrm{sinh}x$; $\dfrac{\mathrm{d}}{\mathrm{d}x}\mathrm{sinh}x = \mathrm{cosh}x$.

(4.2) The wave equation for capillary waves in shallow waters is

$$\frac{\partial^2 h}{\partial t^2} = -\frac{h_0\sigma}{\rho}\frac{\partial^4 h}{\partial x^4}$$

where h_0 is the water depth, σ is the surface tension, ρ is the density of water and $h(x,t)$ is the water displacement at time t and position x. Show that travelling waves of the form

$$h(x,t) = h(u)$$

where $u = x - vt$, are in general NOT solutions for the capillary wave equation above, and explain the reason why it is so. *[5 marks]*

(4.3) Consider the linear wave equation: $\dfrac{1}{v^2}\dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 y}{\partial x^2}$, and prove mathematically that it is linear. *[3 marks]*

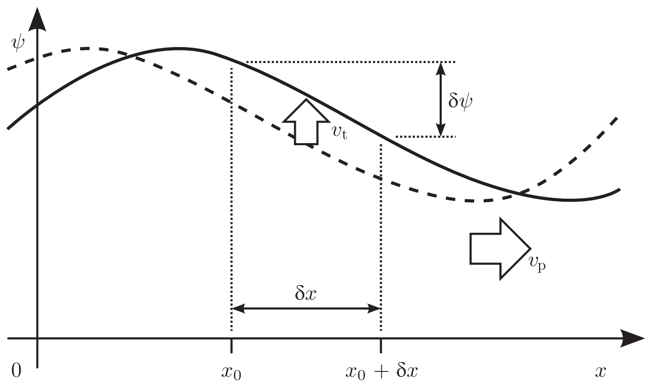
Question 5 [13 marks]

(5.1) A simple harmonic oscillator of amplitude A has a total energy E. Calculate (a) the kinetic energy and (b) the potential energy when the position is one-third of the amplitude. (c) For what values of the position does the kinetic energy equal one-half of the potential energy? *[5 marks]*

(5.2) Consider an element δx of a guitar string undergoing a wave motion like the one shown in the figure below. Use the appropriate assumptions, equations and explanations in order to show that the energy density ϵ is given by

$$\epsilon = T\left(\frac{d\psi}{du}\right)^2,$$

where T is the tension in the string, $u = x - v_p t$ and $\psi = \psi(x,t)$ is the wave displacement at time t and position x. *[8 marks]*



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