UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE

PHYSICS

PHY0037

ADVANCED STATISTICAL MECHANICS

EXAMINATION
JUNE 2016

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EXTERNAL MODERATOR:	Prof R Duvenhage (UP)
TIME: 3 HOURS	**MARKS: 100**

This paper consists of 7 pages.

Please read the following instructions carefully:

1. ANSWER ANY FOUR QUESTIONS

2. No programmable calculators are allowed.

QUESTION 1 **[25]**

(a) It can be readily shown (from its definition) that the grand partition function in statistical mechanics can be expressed in the following form:

$$Z(z,V,T) = \begin{cases} \prod_{\varepsilon} (1 + ze^{-\beta\varepsilon}) & \text{for fermions} \\ \prod_{\varepsilon} \left(\dfrac{1}{1 - ze^{-\beta\varepsilon}} \right) & \text{for bosons} \end{cases}$$

where the symbols z, V, T, β and ε have their usual meaning.

i) Take this as given and derive an expression for the 'q-potential', defined as q = ln (Z), in terms of the same symbols used above.

It can also be shown (from the definition of the grand partition function) that it can be expressed in the form:

$$Z(z,V,T) = \sum_{N=0}^{\infty} [z^N \sum_{\{n_\varepsilon\}}{}^{/} \left(e^{-\beta \sum_{\varepsilon} n_\varepsilon \varepsilon} \right)]$$

where n_ε represents the number of particles with energy ε and the prime on the second summation reflects the condition that the total particle number must correspond to the current value of N in the first summation.

ii) Use this expression, coupled with the result in (i) above, to derive the following expression for the expectation number of the number of particles in the quantum state with energy ε :

$$\langle n_\varepsilon \rangle = [e^{(\varepsilon - \mu)/kT} + a]^{-1}$$

(explain the meaning of the parameter a during your derivation). (12)

(b) Take the following general results that can be obtained from the *grand canonical ensemble* description of a system composed of *N* identical *bosons* as given:

$$\frac{PV}{kT} \equiv \ln \mathbf{Z} = -\sum \ln\left(1 - ze^{-\beta\varepsilon}\right);$$

$$N = \sum_{\varepsilon} \langle n_\varepsilon \rangle = \sum_{\varepsilon} \frac{1}{z^{-1} e^{\beta\varepsilon} - 1},$$

and show that, for a system large enough that the allowed energy levels are very closely spaced relative to the total system energy, we can write

/.... page 3

$$\frac{P}{kT} = -\frac{2\pi(2m)^{3/2}}{h^3}\int_0^\infty \varepsilon^{1/2}\ln\left(1 - ze^{-\beta\varepsilon}\right)\, d\varepsilon - \frac{1}{V}\ln(1-z)\ ;$$

$$\frac{N}{V} = \frac{2\pi(2m)^{3/2}}{h^3}\int_0^\infty \frac{\varepsilon^{1/2}d\varepsilon}{z^{-1}e^{\beta\varepsilon} - 1} + \frac{1}{V}\frac{z}{1-z}\ ,$$

explaining all the terms in these equations in the course of your derivation. (5)

(c) Given the following definition of the so-called 'Bose-Einstein functions':

$$g_\nu(z) \equiv \frac{1}{\Gamma(\nu)}\int_0^\infty \frac{x^{\nu-1}}{z^{-1}e^x - 1}dx = z + \frac{z^2}{2^\nu} + \frac{z^3}{3^\nu} + \ldots..,$$

show that the last two equations expressed in question (b) can be written in the following form as well:

$$\frac{P}{kT} = \frac{1}{\lambda^3}g_{5/2}(z);$$

$$\frac{N - N_0}{V} = \frac{1}{\lambda^3}g_{3/2}(z).$$

Note: Explain the meaning of the λ^3 factor as part of your derivation. You may use the equations $\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$ and $\Gamma(5/2) = \frac{3\sqrt{\pi}}{4}$.

(8)

QUESTION 2 **[25]**

(a) Use appropriate equations appearing elsewhere in this paper, as well as appropriate physical and mathematical arguments, to derive the following equation for a gas of non-interacting bosons:

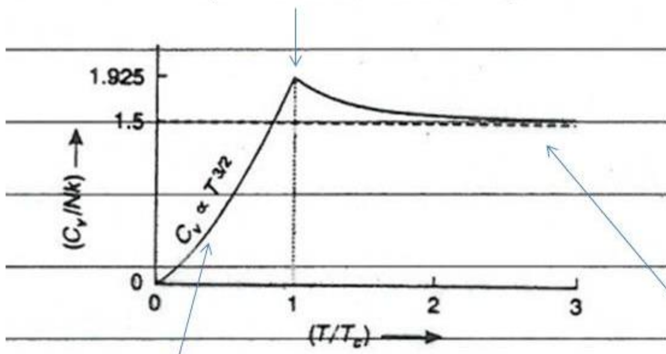
$$T_c = \frac{h^2}{2\pi mk}\left(\frac{N}{V.\zeta(3/2)}\right)^{2/3}.$$

Discuss the physical interpretation of this equation. (9)

[You may refer to the following series expansion which applies for z < 1: $g_{3/2}(z) = z + \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \ldots\ldots.$]

(b) Consider the following diagram, depicting the different regimes of behaviour for the heat capacity (at constant volume) of an ideal boson gas. Provide a *brief explanation* of how each of the stated equations is derived and discuss the physical interpretation of each regime shown:

$$C_V \approx 1.925Nk \quad \text{for} \quad T = T_c$$



$$\frac{C_V}{Nk} = \frac{15}{4}\varsigma(5/2)\left(\frac{v}{\lambda^3}\right) \quad \text{for} \quad T < T_c \qquad\qquad \frac{C_V}{Nk} = \frac{15}{4}\frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{9}{4}\frac{g_{3/2}(z)}{g_{1/2}(z)} \quad \text{for} \quad T > T_c$$

Note: the series expansion for the Bose-Einstein functions: $g_\nu(z) = \sum_{\ell=1}^{\infty} z^\ell \ell^{-\nu} \xrightarrow[1]{z} \sum_{\ell=1}^{\infty} \ell^{-\nu} \equiv \varsigma(\nu)$

can be 'inverted' as follows: $z = \sum_\gamma a_\gamma \left(\frac{N}{V}\lambda^3\right)^\gamma$ (where γ is a positive integer) for small values of λ.

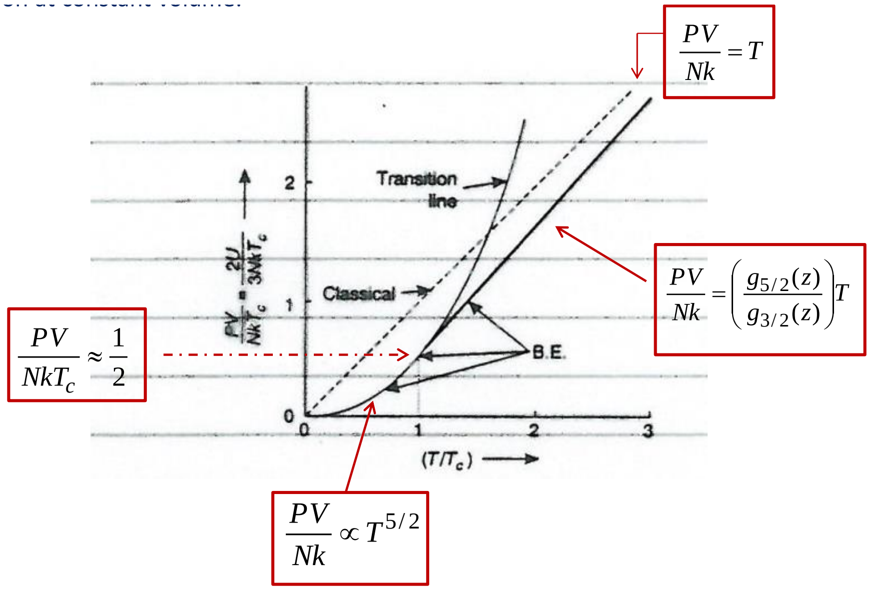
(11)

(c) Prove the following result for the Bose-Einstein functions (you may use the series expansion appearing in part (b) above):

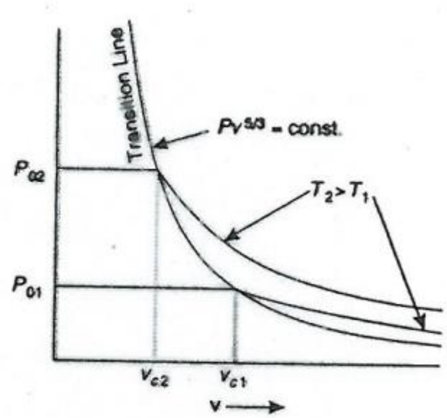
$$z\frac{\partial}{\partial z} g_\nu(z) = g_{\nu-1}(z) \tag{5}$$

QUESTION 3 **[25]**

(a) The following plot shows the relationship between pressure and temperature, for an ideal boson gas. There are 4 equations shown in blocks. Derive each of them, while also commenting on the physical meaning of the shape of the plot across the temperature domain. (10)



(b) Repeat the instructions in part (a), but now for the equivalent of Boyle's Law (i.e the relationship between pressure and volume when temperature remains constant) for an ideal Bose gas, as depicted in the figure shown here:



(10)

(c) The following equations express the thermodynamic parameters of an ideal Fermi gas in terms of the fugacity:

$$\frac{P}{kT} = +\frac{2\pi}{h^3}(2m)^{3/2}\int_0^\infty \varepsilon^{1/2}\ln(1 + ze^{-\beta\varepsilon})d\varepsilon$$

$$\frac{N}{V} = \frac{2\pi}{h^3}(2m)^{3/2}\int_0^\infty \frac{\varepsilon^{1/2}d\varepsilon}{z^{-1}e^{\beta\varepsilon}+1}.$$

These equations can be expressed in terms of the Fermi-Dirac functions, as follows:

/.... page 6

$$\frac{P}{kT} = \frac{1}{\lambda^3} f_{5/2}(z); \qquad \frac{N}{V} = \frac{1}{\lambda^3} f_{3/2}(z).$$

Show that $PV = \dfrac{2}{3}U$ for such a gas. (5)

QUESTION 4 **[25]**

(a) Make appropriate assumptions about *photons* and then derive the following expressions that apply to an 'ideal gas of photons' (i.e. a radiation field):

i) $$PV = \frac{8\pi^5 V}{45h^3c^3} k^4T^4$$

ii) $$U = \frac{8\pi^5 V}{15h^3c^3} k^4T^4$$

(12)

(b) Start with the generic equations for pressure and particle density of an ideal Fermi gas:

$$P = \frac{4\pi}{3h^3}\int_0^\infty \frac{1}{1+z^{-1}e^{\beta\varepsilon}}\left(p\frac{d\varepsilon}{dp}\right)p^2 dp; \qquad \frac{N}{V} = \frac{4\pi}{h^3}\int_0^\infty \frac{1}{1+z^{-1}e^{\beta\varepsilon}}\, p^2 dp\,,$$

then *briefly* explain (*qualitative* explanations will do) why the electrons in a white dwarf star must be highly degenerate and also highly relativistic, and then derive the following expression for the degenerate electron pressure in a white dwarf:

$$P = \frac{\pi m^4 c^5}{3h^3} A(x)\,.$$

[where $\theta = \sinh^{-1}(p/mc)$, $x = \sinh(\theta_{Fermi})$ and $A(x) = 8\int_0^{\theta_{Fermi}} \sinh^4 \theta d\theta$]. (13)

QUESTION 5 **[25]**

Consider the diamagnetic effect acting on a gas of fermions placed in a uniform magnetic field, as first examined by Landau. Start by choosing a reference frame with a *z*-axis parallel to the field direction. Assume that the fermions are trapped inside a box with a width L_x in the *x*-direction and a width L_y in the *y*-direction. It has been shown that the quantised energy levels accessible to a fermion with mass *m*, charge *e* and momentum $\vec{p}$ in this scenario are given by:

$$\varepsilon_j = \frac{eB\hbar}{mc}(j+1/2)+\frac{p_z^2}{2m}, \ (j = 0, 1, 2, ...).$$

You may use this result as given. Now:

i) Show that the total number of accessible states corresponding to any particular value of j is given by:

$$L_x L_y\left(\frac{eB}{hc}\right). \qquad (6)$$

ii) Show that the q-potential (i.e. ln Z) of the system takes the following form in the classical limit ($T \to \infty$):

$$\ln Z = \frac{eVB}{h^2 c}(2m\pi kT)^{1/2}\left(2\sinh^{-1}\left(\frac{eB\hbar}{2mckT}\right)\right)^{-1}. \qquad (11)$$

[Hint: $\ln(1+x) \approx x$ when x is small]. You may use the following standard integral in your derivation:

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\pi/a}$$

iii) It can easily be shown that the magnetisation due to the diamagnetic effect can be written in the form:

$$M = \beta^{-1}\left\langle\frac{\partial \ln Z}{\partial B}\right\rangle_{z,V,T}.$$

Use the result in ii), to transform this to the form:

$$M = -N\mu_{eff}L(x)$$

where $\mu_{eff} = \frac{e\hbar}{2mc}$, and explain what $L(x)$ represents. What is the significance of the minus sign in this equation? (8)

USEFUL INFORMATION:

Various valid forms of the density of states, depending on the physical context:

$$a(p)dp = g\frac{V}{h^3}4\pi p^2 dp \qquad a(\varepsilon)d\varepsilon = \frac{8\pi V}{h^3c^3}\varepsilon^2 d\varepsilon \qquad a(\varepsilon)d\varepsilon = \left(\frac{2\pi V}{h^3}\right)(2m)^{3/2}\varepsilon^{1/2}d\varepsilon$$

A common standard integral: $\displaystyle\int_0^\infty \frac{x^3}{e^x-1}dx = \frac{\pi^4}{15}$

END OF EXAM PAPER