Question 1 [24 marks]

(1.1) Consider a thin, flexible string of mass per unit length ρ and subject to a tension T. By considering the net force acting on an element of the string, derive the wave equation governing its transverse motion

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2},$$

where $y(x,t)$ is the wave displacement at time t and position x, and $\frac{T}{\rho} = v^2$. Please note: Explain all steps and symbols used, and make use of a diagram. *[6 marks]*

(1.2) Show that travelling waves of the form

$$y(x,t) = y(u),$$

where $u = x - vt$, may be solutions of the linear wave equation above, and find the two values of v that make them so. *[6 marks]*

(1.3) Show that the linear wave equation in question (1.1) has a sinusoidal wave solution of the form

$$y(x,t) = a_0 \cos(kx - \omega t + \phi),$$

and find the relationship between the parameters v, k and ω which makes it so. *[3 marks]*

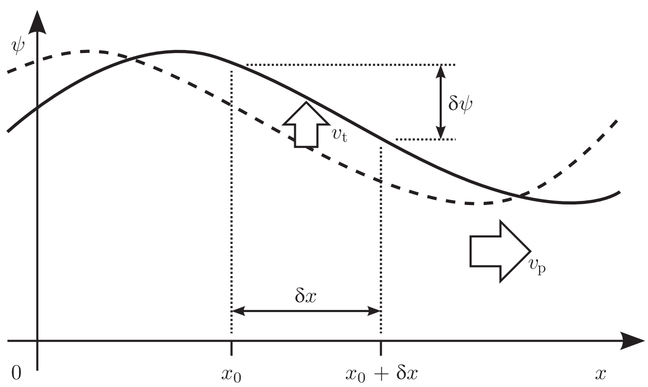
(1.4) Consider the wave function $y(x,t) = x^2 + v^2t^2$.
(a) Show that $y(x,t)$ is a solution of the linear wave equation in question (1.1). *[2 marks]*
(b) Show that $y(x,t)$ can be written as $y(x,t) = f(x+vt) + g(x-vt)$ and determine the functional forms of f and g. *[2 marks]*
(c) Repeat parts (a) and (b) for the function $y(x,t) = \sin(x)\cos(vt)$. *[5 marks]*

Question 2 [15 marks]

(2.1) Consider an element δx of a guitar string undergoing a wave motion like the one shown in the figure below. Use the appropriate assumptions, equations and explanations in order to show that the energy density ϵ is given by

$$\epsilon = T\left(\frac{d\psi}{du}\right)^2,$$

where T is the tension in the string, $u = x - v_p t$ and $\psi = \psi(x,t)$ is the wave displacement at time t and position x. *[7 marks]*



2.2 Write down the differential equation that describes the motion of a simple harmonic oscillator with spring constant k and mass m. Show that a generic exponential function can be a solution of this equation <u>if and only if</u> it is a complex exponential. *[3 marks]*

2.3 Show that this expression: $x(t) = C_1 e^{(+i\omega t)} + C_2 e^{(-i\omega t)}$ (where $\omega = \sqrt{k/m}$) is a solution of the simple harmonic differential equation you wrote in question 2.2 if and only if $C_1 = C_2$. *[5 marks]*

Question 3 [14 marks]

The wave equation governing the transverse motion for a skipping rope is the following:

$$M\frac{\partial^2 y}{\partial t^2} = W\frac{\partial^2 y}{\partial x^2} - \gamma\frac{\partial y}{\partial t},$$

where M is the linear mass density, W is the tension in the string, γ is a damping coefficient and $y(x,t)$ is the wave displacement at time t and position x.

(3.1) Show that the following complex exponential waveform

$$y(x,t) = a \exp i(kx - \omega t)$$

(where i is the imaginary unit defined as $i = \sqrt{-1}$, and a is a constant) can be a solution of the wave equation above, provided that the following condition holds: $k^2 W = \omega^2 M + i\gamma\omega$. *[4 marks]*

(3.2) Assuming ω to be *complex* and k to be *real*, solve the equation $k^2 W = \omega^2 M + i\gamma\omega$ for ω and calculate the expressions for the real and the imaginary part of ω assuming that $\gamma \ll k\sqrt{WM}$. At the end of this derivation show that the complex exponential waveform above can be re-written as

$$y(x,t) = a \exp i(kx - \omega_0 t) \exp(-\frac{\gamma}{2M}t)$$

where $\omega_0 = k\sqrt{\frac{W}{M}}$. *[10 marks]*

Question 4 [15 marks]

(4.1) The capillary-gravity wave equation for waves in shallow water is given by the following:

$$\frac{\partial^2 h}{\partial t^2} = g h_0 \frac{\partial^2 h}{\partial x^2} - \frac{h_0 \sigma}{\rho}\frac{\partial^4 h}{\partial x^4}$$

where g is the acceleration of gravity, h_0 is the water depth, σ is the surface tension, ρ is the density of water and $h(x,t)$ is the water displacement at time t and position x. Show that a sinusoidal waveform $h(x,t) = h_1 \sin(ku)$ (where $u = x - vt$) may be a solution of this wave equation, and find the expression for v as a function of k that makes it so. *[5 marks]*

(4.2) The Korteweg - de Vries wave equation:

$$\frac{\partial \psi}{\partial t} + 6\psi\frac{\partial \psi}{\partial x} + \frac{\partial^3 \psi}{\partial x^3} = 0$$

describes the motion of a soliton, i.e. a single wave pulse ψ that maintains its shape while it travels at constant speed. Show that the travelling wave pulse (with propagation speed c) given by the following:

$$\psi(x,t) = 2\alpha^2 \mathrm{sech}^2[\alpha(x - ct)]$$

may be a solution of the Korteweg - de Vries wave equation, and find the particular values of the parameter α that make it so. *[10 marks]*

Please take note of the following:
$\mathrm{sech}x = \frac{1}{\cosh x}$; $\frac{\mathrm{d}}{\mathrm{d}x}\cosh x = \sinh x$; $\frac{\mathrm{d}}{\mathrm{d}x}\sinh x = \cosh x$.

Question 5 [22 marks]

(5.1) Explain what it means mathematically for a wave equation to be defined as *linear*. *[2 marks]*

(5.2) Consider the linear wave equation:

$$\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2},$$

and prove mathematically that it is indeed linear. *[4 marks]*

(5.3) Consider the following wave equation:

$$\frac{\partial y}{\partial t} + y\frac{\partial y}{\partial x} + \frac{\partial^3 y}{\partial t^3} = 0\,,$$

where the wave displacement at time t and position x is given by $y(x,t)$. Is this wave equation linear or not? Substantiate your answer mathematically and carefully explain your conclusion. *[7 marks]*

(5.4) A simple harmonic oscillator of amplitude A has a total energy E. Calculate (a) the kinetic energy and (b) the potential energy when the position is one-third of the amplitude. (c) For what values of the position does the kinetic energy equal one-half of the potential energy? (d) Are there any values of the position where the kinetic energy is greater than the maximum potential energy? Explain your answer. *[9 marks]*

END of QUESTION PAPER