UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MODULE	**PHY0017**
CAMPUS	**APK**
EXAM	**SUPPLEMENTARY JULY 2016**

DATE: 27/07/2016 **SESSION: 09:00**

ASSESSOR(S): **DR BP DOYLE**

EXTERNAL MODERATOR: **PROF R DE MELLO KOCH (WITS)**

DURATION: 3 HOURS **MARKS: 80**

NUMBER OF PAGES: 4 PAGES INCLUDING THIS ONE AND THE INFORMATION SHEET

INSTRUCTIONS: **Answer all the questions.**

Explanations are necessary!

Question 1 [22]

(1a) Derive the first order correction to the energy of the eigenvalues in time-independent perturbation theory. [7]

(1b) Consider a delta-function bump in the centre of the infinite square well:

$H' = \alpha\delta(x - a/2)$,

where a is the width of the well and δ is a constant. Find the first-order correction to the allowed energies. Explain why the energies are not perturbed for even n. [4]

(1c) Show that in two-fold degenerate perturbation theory the energies are given by:

$$E^1_\pm = \frac{1}{2}\left[W_{aa} + W_{bb} \pm \sqrt{(W_{aa} - W_{bb})^2 + 4|W_{ab}|^2}\right],$$

where $W_{ij} \equiv \langle\psi^0_i|H'|\psi^0_j\rangle$, $\quad i, j = a, b$. and H' is the perturbation to the Hamiltonian. [7]

(1d) Show that the spin-orbit interaction perturbation to the Hamiltonian ($H = \left(\frac{e^2}{8\pi\epsilon_0}\right)\frac{1}{m^2c^2r^3}\mathbf{S}\cdot\mathbf{L}$) does not commute with the orbital angular momentum. (Hint: you will need the commutation relations for angular momentum). [4]

Question 2 [10]

(2a) Derive Ehrenfests theorem for a time independent operator. [4]

(2b) Use this result to derive the quantum mechanical version of the classical equation $\frac{\partial\mathscr{H}}{\partial p_i} = \dot{q}_i$, where q_i is a position coordinate. [6]

Question 3 [13]

(3a) Use the path integral recipe to show that a 1 g particle will follow a classical trajectory, whereas an electron's trajectory is not well defined. (Hint: consider two paths, one of which is the classical one). [6]

(3b) Show that the integral that results when one exactly evaluates the free-particle propagator is,

$$\lim_{N\to\infty} A' \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\left[-\sum_{i=0}^{N-1}\frac{(y_{i+1} - y_i)^2}{i}\right] dy_1\cdots dy_{N-1}$$

where

$$A' = A\left(\frac{2\hbar\varepsilon}{m}\right)^{(N-1)/2}.$$

You should break the time interval up into steps and at each time insert a complete set of positions. [7]

Question 4 [21]

(4a) Determine the translation operator $T(\varepsilon)$ to first order in ε, in the **passive transformation picture**. [7]

(4b) Show that time translational invariance leads to the conservation of energy. [6]

(4c) Show that $\Pi = \Pi^{-1}$ and that the eigenvalues of Π are ± 1, where Π is the parity operator. You may assume that $\Pi^2 = I$. [3]

(4d) Show that the requirement for a quantum system to have time-reversal symmetry is that the Hamiltonian must be real. [5]

Question 5 [14]

(5a) The action of the rotation operator on position eigenkets is defined as,

$$U[R]\,|x, y\rangle = |x\cos\phi_0 - y\sin\phi_0, x\sin\phi_0 + y\cos\phi_0\rangle$$

(i) To first order, prove that,

$$\langle x, y|I - \frac{i\varepsilon_z L_z}{\hbar}|\psi\rangle = \psi(x + y\varepsilon_z, y - x\varepsilon_z)$$ [6]

(ii) Now show that this leads to,

$$L_z = XP_y - YP_x$$ [4]

(5b) Write down the transformation equation for an infinitesimal rotation of X in the passive picture. Use this to show that,

$$[X, L_z] = -i\hbar Y$$ [4]

END of QUESTIONS

INFORMATION SHEET

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$$

$[r_i, p_j] = -[p_i, r_j] = i\hbar\delta_{ij} \quad , \quad [r_i, r_j] = [p_i, p_j] = 0$

$[L_x, L_y] = i\hbar L_z \quad , \quad [S_x, S_y] = i\hbar S_z$

For operators: $[\Omega, \Lambda\theta] = \Lambda[\Omega,\theta] + [\Omega,\Lambda]\theta$ and $[\Lambda\Omega, \theta] = \Lambda[\Omega,\theta] + [\Lambda,\theta]\Omega$

Ehrenfest's theorem: $\frac{d}{dt}\langle\Omega\rangle = \left(\frac{-i}{\hbar}\right)\langle[\Omega, H]\rangle$

$$\int \sin^2(ax)\, dx = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int \sin(mx)\sin(nx)\, dx = \left(\frac{\sin[(m-n)x]}{2(m-n)} - \frac{\sin[(m+n)x]}{2(m+n)}\right)$$

$\hbar = 1.05457 \times 10^{-34} J \cdot s$

Wavefunctions of the infinite square well: $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$

$$a = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} = \frac{\hbar}{\alpha m_e c} = 5.29177 \times 10^{-11} m$$

$m_e = 9.10938 \times 10^{-31} kg$

$m_p = 1.67262 \times 10^{-27} kg$

$$\delta(x - x') \equiv \lim_{\Delta\to 0}\frac{1}{(\pi\Delta^2)^{1/2}}\exp\left[\frac{(x-x')^2}{\Delta^2}\right]$$

Fresnel integrals: $\int_0^\infty \cos(t^2)dt = \int_0^\infty \sin(t^2)dt = \sqrt{\frac{\pi}{8}}$

Gaussian integral: $\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2}\, dx = \frac{1}{2\alpha}\left(\frac{\pi}{\alpha}\right)^{1/2}$

$$e^{-ax} = \lim_{N\to\infty}\left(1 - \frac{ax}{N}\right)^N$$

Taylor series: $f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$

Taylor series for e^x: $1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

$$\int \mathscr{D}[x(t)] = \lim_{\substack{\varepsilon\to 0\\ N\to\infty}}\frac{1}{B}\int_{-\infty}^{\infty}\int\int\cdots\int_{-\infty}^{\infty}\frac{dx_1}{B}\cdot\frac{dx_2}{B}\cdots\frac{dx_{N-1}}{B} \quad \text{where} \quad B = \left(\frac{2\pi\hbar\varepsilon i}{m}\right)^{1/2}$$

$$\int_a^b \langle x|x'\rangle\langle x'|f\rangle\, dx' = \langle x|I|f\rangle = \langle x|f\rangle$$

For the hydrogen atom: $\left\langle\frac{1}{r}\right\rangle = \frac{1}{n^2 a}$; $\left\langle\frac{1}{r^2}\right\rangle = \frac{1}{(l^2+1/2)n^3a^2}$; $\left\langle\frac{1}{r^3}\right\rangle = \frac{1}{l(l+1/2)(l+1)n^3a^3}$

Bohr energies of hydrogen: $E_n = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2} = \frac{E_1}{n^2}$