1. Let $x = (\xi_j)$ be a sequence of real numbers.
   (a) Show that if $x \in l^1$ then $x \in l^2$. (4)
   (b) With the assumption that $x \in l^1$ show that (6)

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1,$$

   where $\|\cdot\|_\infty$, $\|\cdot\|_2$, and $\|\cdot\|_1$ are the l^∞, l^2, and l^1 norms respectively.

2. Consider the Banach space $X = C[-1,1]$. Define $T : X \to X$ to be the operator given by

$$(Tx)(t) = e^t x(t) \text{ for each } x \in C[-1,1],$$

   and define $S : X \to X$ to be

$$(Sx)(t) = (2 - t^2) \int_{-1}^{t} x(s)\, ds \text{ for each } x \in C[-1,1]$$

   (a) Explain why T is well-defined, and why S is well-defined. That is, explain why the image of an element $x \in X$, under the operators T and S does in fact belong to X. (4)
   (b) Show that both T and S are bounded linear operators on X, and then calculate only $\|T\|$. (6)

3. Show that the dual space of c_0 is (isometrically isomorphic to) l^1. (8)

4. (a) Let X be an inner product space. Show that any set of m mutually orthogonal non-zero vectors $\{x_1, x_2, \ldots, x_m\}$ is a linearly independent set. (4)
   (b) Show that (6)

$$Y = \{x : x = (\xi_j) \in l^2, \xi_{2n} = 0, n \in \mathbb{N}\}$$

   is a closed vector subspace of l^2 and then find the orthogonal complement $Y^\perp$. If

$$x = (\underbrace{1, 1, \ldots, 1}_{10 \text{ places}}, 0, 0, 0, 0, \ldots),$$

   then find the unique vector $y \in Y$ that minimizes the distance between x and Y, and then calculate this distance.
   (c) Let X be an inner product space and suppose $x, y \in X$ are linearly independent vectors in X with $\|x\| = 1 = \|y\|$. Show that (4)

$$\|tx + (1-t)y\| < 1 \text{ for all } 0 < t < 1.$$

Total:42