UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE

DEPARTMENT OF MATHEMATICS

MODULE	MAT0057 SET THEORY (HONOURS)
CAMPUS	APK
EXAM	JUNE EXAM 2016

DATE:	9 JUNE 2016
SESSION:	08:30
ASSESSOR(S):	DR A. CRAIG
EXTERNAL MODERATOR:	PROF. T.A. BATUBENGE UNISA
DURATION:	3 HOURS
MARKS:	75

NUMBER OF PAGES: 1 + 3 PAGES

INSTRUCTIONS:
1. ANSWER ALL THE QUESTIONS.
2. USE THE ANSWER BOOKS PROVIDED.
3. QUESTIONS CAN BE ANSWERED IN ANY ORDER.
4. SHOW ALL CALCULATIONS & MOTIVATE ALL ANSWERS.
5. GOOD LUCK.

Question 1 [2 marks]

Prove that there is no set to which every set belongs.

Question 2 [3 marks]

(a) Find $\bigcap A$ if $A = \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$. (1)

(b) Calculate $\bigcup\bigcup B$ if $B = \{\, \{\{a\}\}, \{\{b\}, \{b, \{c\}\}\} \,\}$. (2)

Question 3 [4 marks]

(a) Let x, y be sets. Give the definition of the ordered pair $\langle x, y \rangle$. (1)

(b) Let a, b be sets. Show that if $\langle x, y \rangle = \langle a, b \rangle$ then $x = a$ and $y = b$. (3)

Question 4 [6 marks]

Let $A = \{w, x, y, z\}$ and consider $R = \{\langle w, z \rangle, \langle y, y \rangle, \langle z, x \rangle, \langle w, w \rangle\}$.

(a) Define a set S such that $R \cup S$ is an equivalence relation on A. (3)

(b) Calculate $[w]_{R \cup S}$. (1)

(c) Let $T \subseteq \omega \times \omega$ be given by

$$xTy \iff y > x + 1$$

Is T a linear ordering on ω? Justify your answer. (2)

Question 5 [6 marks]

Let A be a set and R a binary relation on A.

(a) Let $\mathcal{B}$ be a non-empty set of transitive relations on A. Prove that $\bigcap \mathcal{B}$ is a transitive relation on A. (3)

(b) Now prove that there exists a least (smallest in terms of set inclusion) transitive relation containing R. (This relation is called the *transitive closure* of R.) (3)

Question 6 [2 marks]

Let F be a one-to-one function. Show that for any sets A and B we have $F[\![A]\!] \cap F[\![B]\!] \subseteq F[\![A \cap B]\!]$.

Question 7 [7 marks]

(a) Define the successor of a set a and state what it means for a set B to be inductive. (2)

(b) Give the definition of a natural number. (1)

(c) Give two definitions of a transitive set and show that they are equivalent. (4)

Question 8 [6 marks]

Let $\sigma(n) = n^+$ for all $n \in \omega$. Prove that $\langle \omega, \sigma, 0 \rangle$ is isomorphic to any Peano system $\langle N, S, e \rangle$.

Question 9 [4 marks]

Show that for any natural numbers m and n:

$$m \in n \quad \text{if and only if} \quad m^+ \in n^+$$

Question 10 [5 marks]

Consider the following extract from the proof of the Well Ordering of ω and answer the questions that follow.

Assume that A is a subset of ω without a least element. We will show that $A = \emptyset$. Define a set $B \subseteq \omega$ as follows:

$$B = \dots$$

We will show that B is inductive. Clearly $0 \in B$. Let $k \in B$ and consider $k^+ \dots$

(a) Define the set B. (1)

(b) Prove that $k^+ \in B$. (2)

(c) Explain why B being inductive implies that $A = \emptyset$. (2)

Question 11 [2 marks]

Prove that no set is equinumerous to its powerset.

Question 12 [3 marks]

If $A \subset \omega$, is it true that $\bigcup A \in \omega$? If so, prove it. If not, give a counterexample.

Question 13 [6 marks]

(a) State the Schröder–Bernstein Theorem (both parts). (2)

(b) Prove that the function
$$f : \omega \times \omega \to \omega, \quad \langle n, m \rangle \mapsto 2^n.3^m$$
is injective. (2)

(c) Use parts (a) and (b) to prove that $\omega \times \omega \approx \omega$. (2)

Question 14 [4 marks]

(a) Let A be a set. Explain what it means for $B \subseteq A$ to be a *chain*. (2)

(b) When successfully applied to a set A, Zorn's Lemma guarantees the existence of a maximal element. What happens in the case that A is the empty set? (2)

Question 15 [7 marks]

(a) State two forms of the Axiom of Choice other than Zorn's Lemma. (4)

(b) Show that one of your stated versions of the Axiom of Choice implies the other. (3)

Question 16 [3 marks]

(a) Give the definition of a countable set. (1)

(b) Let A, B be countable sets. Show that $A \cup B$ is countable. (2)

Question 17 [5 marks]

(a) State and prove the Absorption Law for Cardinal Arithmetic. (3)

(b) Let $2 \leqslant \kappa \leqslant \lambda$. Show that if λ is infinite then $\kappa^\lambda = 2^\lambda$. (2)