UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE
FAKULTEIT NATUURWETENSKAPPE

DEPARTMENT OF MATHEMATICS / DEPARTEMENT WISKUNDE

MODULE	**MAT3B20** INTRODUCTORY ABSTRACT ALGEBRA INLEIDENDE ABSTRAKTE ALGEBRA
CAMPUS KAMPUS	**APK** APK
EXAM EKSAMEN	**NOVEMBER 2016**

DATE
DATUM 23/11/2016

SESSION
SESSIE 8:30–11:00

ASSESSOR(S) DR. E. JOUBERT

INTERNAL MODERATOR
INTERNE MODERATOR PROF. L. VAN WYK

EXTERNAL MODERATOR
EKSTERNE MODERATOR

DURATION 2,5 HOURS
TYDSDUUR 2,5 UUR

MARKS 85
PUNTE 85

SURNAME AND INITIALS
VAN EN VOORLETTERS ______________________________

STUDENT NUMBER
STUDENTENOMMER ______________________________

CONTACT NR
KONTAK NO ______________________________

NUMBER OF PAGES: 3 PAGES INCLUDING THE COVER
AANTAL BLADSYE: 3 BLADSYE INSLUITEND DIE DEKBLAD

INSTRUCTIONS:
1) **ANSWER EACH QUESTION IN THE ANSWER BOOKS PROVIDED. IF NECESSARY USE THE BACK OF THE PAGES AND INDICATE IT CLEARLY.**
2) **ONLY NON-PROGRAMMABLE CALCULATORS MAY BE USED.**

INSTRUKSIES:
1) BEANTWOORD AL DIE VRAE IN DIE GEGEWE ANTWOORDBOEK. AS NODIG GEBRUIK DIE AGTERKANT VAN DIE BLADSYE EN DUI DIT DUIDELIK AAN.
2) SLEGS NIE-PROGRAMMEERBARE SAKREKENAARS MAG GEBRUIK WORD.

QUESTION 1 **[20]**

1.1 Let G and $\overline{G}$ be groups and let ϕ be a function from G to $\overline{G}$. Answer the following questions:

1.1.1 Provide a detailed definition of when ϕ is a **homomorphism**. In addition, define $Ker\phi$. [2]
1.1.2 Prove the following: If $\phi(g) = g'$, then $\phi^{-1}(g') = \{x \in G \mid \phi(x) = g'\} = gKer\phi$. [3]
1.1.3 Prove that $Ker\phi$ is a normal subgroup of G. [3]

1.2 Find all homomorphisms from Z_{10} to Z_{15}. [4]

1.3 Let H and K be two groups, where the identity element in K is denoted by e. Answer the following questions:

1.3.1 Prove that the function $\phi : H \oplus K \to K$, defined by $\phi((h, k)) = k$, where $h \in H$ and $k \in K$, is a homomorphism. [2]
1.3.2 Find $Ker\phi$. [2]
1.3.3 Prove that $\phi(H \oplus K) = K$. [2]
1.3.4 Hence, or otherwise, prove that $H \oplus K \;/\; (H \oplus \{e\}) \simeq Z$. [2]

QUESTION 2 **[18]**

2.1 Consider the group $U(10) \oplus Z_8$. Answer the following questions:

2.1.1 How many elements of order 4 are there in $U(10) \oplus Z_8$? [3]
2.1.2 Is $U(10) \oplus Z_8$ cyclic? Explain your answer. [2]
2.1.3 Are $U(10) \oplus Z_8$ and Z_{32} isomorphic? Explain your answer. [1]

2.2 Let G be a group and let $\phi : G \to G$ be defined by $\phi(g) = g^{-1}$, for all $g \in G$. Answer the following questions:

2.2.1 Show that if G is **Abelian**, then ϕ is an **automorphism**. [2]
2.2.2 Show that if G is an **automorphism**, then G is **Abelian**. [3]

2.3 Let G be an Abelian group of order n. Suppose G has at least **3** elements of order **3**, and let $a, b \in G$ be elements both of which have order **3**. Answer the following questions:

2.3.1 Define the set $G/\langle a \rangle$. [1]
2.3.2 Find $|G/\langle a \rangle|$. [1]
2.3.3 If $b \notin \langle a \rangle$, show that $|b\langle a \rangle| = 3$. [3]
2.3.4 Prove that 9 divides n. [2]

QUESTION 3 **[22]**

3.1 Consider the set R of integers, where $a, b \in R$. Define the following **multiplicative** (given by $\otimes$) and **additive** (given by $\div$) operations on R by: $a \otimes b = a$ and $a \div b = a + b + 1$.

3.1.1 Prove that R is an Abelian group under $\div$. [3]
3.1.2 Prove that R is associative under $\otimes$. [2]
3.1.3 Is R a ring under $\otimes$ and $\div$? Motivate your answer. [2]

3.2 Let R be a ring and let $a, b \in R$. Prove the following multiplicative property of R: $(-a)(-b) = ab$. [2]

3.3 Let R be a ring. Define $S = \{x \in R \mid ax = xa \text{ for all } a \in R\}$. Show that S is a subring of R.[3]

3.4 Let R be a ring and let A, B and C be subrings of R. Prove that if $A \subseteq B \cup C$, then $A \subseteq B$ or $A \subseteq C$. [3]

3.5 Let F be a finite integral domain. Prove that F is a field. [4]

3.7 Let R be a ring with unity 1. If the product of any pair of nonzero elements of R is nonzero, prove that $ab = 1$ implies that $ba = 1$. [3]

QUESTION 4 **[25]**

4.1 State the definition of when a subset A of a ring R is an ideal. [2]

4.2 Let A and B be two ideals of a ring R. Prove that $AB \subseteq A \cap B$. [3]

4.3 Suppose that R is a ring and A is a subring of R. Prove that the set $H = \{r + A \mid r \in R\}$ is a ring under the operations $(r + A) + (s + A) = s + r + A$ and $(r + A) \times (s + A) = rs + A$, if and only if A is an ideal of R. [6]

4.4 Let R and S be two rings. Answer the following questions:

4.4.1 If ϕ is a homomorphism from R to S, then prove the **First Isomorpism Theorem for Rings.** [4]
4.4.2 Let ϕ be a ring homomorphism from a ring R to a ring S. Show that if R has a unity 1, $S \neq \{0\}$, and ϕ is onto, then $\phi(1)$ is the unity of S. [3]
4.4.3 Let a and b be two nonzero positive integers. Prove that $\phi : Z \oplus Z \to Z_a \oplus Z_b$, given by $\phi((x, y)) = (x \bmod a, y \bmod b)$, is a ring homomorpism from the ring $Z \oplus Z$, to the ring $Z_a \oplus Z_b$.[3]
4.4.4 Find $Ker\phi$. [2]
4.4.5 Can one say that $Z \oplus Z/(\langle a \rangle \oplus \langle b \rangle)$ is ring isomorphic to $Z_a \oplus Z_b$? Motivate your answer. [2]