UNIVERSITY VAN JOHANNESBURG

UNIVERSITEIT
VAN
JOHANNESBURG

FACULTY OF SCIENCE

MATHEMATICS (APK)

MAT3A20

DISCRETE MATHEMATICS
EXAM
JUNE 2016

EXAMINER	**Prof W E Conradie**
EXTERNAL EXAMINER	**Dr R Kellerman (UP)**
TIME: 150 MINUTES	**80 MARKS**

SURNAME AND INITIALS: ....

STUDENT NUMBER:...

TEL NO: ....

Please read the following instructions carefully:

1. Answer all questions.
2. Write out all calculations (steps).
3. Questions are to be answered on the question paper in the space provided. If you do not have enough space to write an answer, complete it on the back of the previous page. Indicate this clearly.
4. Non-programmable calculators are allowed.
5. This paper consists of **1 + 12** pages.

1. Let A, B and C be arbitrary formulas of propositional logic.
   Prove that if A is satisfiable and $A \models B$, then $B \vee C$ is satisfiable. [3]

2. Negate the formula $\neg(p \to q) \to (\neg r \wedge s)$, and import the negation to stand only directly in front of propositional variables. [3]

3. Consider the following first-order formula:

$$\forall \epsilon(\epsilon > 0 \to \exists \delta \forall x(|x - c| < \delta \to |f(x) - f(c)| < \epsilon)).$$

   (a) What does the formula say about the function f? [2]

   (b) What is the scope of the quantifier $\forall x$ in the formula? [1]

   (c) Is the formula clean? Motivate. [1]

4. Formalize the following argument in first-order logic. Introduce suitable predicate symbols as needed. [3]

$$\frac{\text{All tigers are carnivores.} \quad \text{No carnivore likes pumpkin.}}{\text{Some tigers do not like pumpkin.}}$$

5. Let Q be a unary predicate symbol and g a binary function symbol. Consider the formula $\forall x \forall y (Q(x) \land Q(y) \to Q(g(x,y))$. Show that this formula is not logically valid by providing a first-order structure in which it is false. Remember to specify the domain of the structure as well as the interpretation of Q and g therein. Motivate fully. [3]

6. Let C be the formula $\forall x(\neg(x > z) \vee \exists y(y \times y = x))$, let $\mathcal{R}$ be the structure of real numbers and v a variable assignment on $\mathcal{R}$ such that $v(x) = 2$, $v(y) = 3$ and $v(z) = 4$. Which player, Verifier or Falsifier, has a winning strategy in the game $(\mathcal{R}, v, C)$? Describe the strategy completely. [3]

7. Let P and Q be unary predicate symbols and let R be a binary relation symbol. Use semantic tableau to determine whether [6]

$$\forall x \forall y (P(x) \land Q(y) \to R(x, y)), \exists x P(x), \exists x Q(x) \models \exists x \exists y R(x, y).$$

Section B: Combinatorics 25 Marks

1. Consider the sequences of length 12, in which each position is an A, B, C or D. For example, BCDAACBDBABC is such a sequence.

   (a) How many such sequences are there? [2]

   (b) How many such sequences are there in which each of the letters A, B, C and D occurs at least once? [4]

   (c) How many such sequences are there in which there are exactly 3 As, 2 Bs, 4 Cs and 3 Ds? [2]

   (d) How many of the sequences in (c) have no As standing directly next to each other? [3]

2. The Blue Lagoon Waffle House offers k different types of waffle. They are running a special promotion in which customers get a discount if they order 8 waffles.

   (a) How many different orders for 8 waffles can be placed? [2]

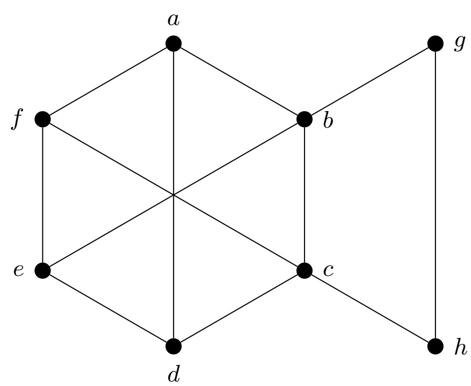
   (b) After 500 orders have been placed, the manager of Blue Lagoon Waffle House knows, without having looked at the orders, that the same order was placed at least 4 times. What is the largest value of k for which this is possible? [3]

3. Find the general solution of the recurrence relation $a_n = 2\sqrt{3}a_{n-1} - 4a_{n-2}$. [4]

4. Solve the recurrence relation given by $a_n = -2a_{n-1} + (n+1)$ with initial condition $a_0 = 3$.

[5]

1. Consider the graph G pictured below, and answer the questions that follow:



   (a) Is G bi-partite? If yes, identify the partite sets. If no, motivate why not. [3]

   (b) Is G planar? If yes, draw it as a plane graph. If no, motivate why not. [3]

2. Suppose that G is a tree such that the sum of the degrees of its vertices is 50. How many vertices does G have? [3]

3. (a) Prove that if G is a non-trivial, connected graph in which every vertex is even, then G is Eulerian.

[6]

(b) Give an example of a connected graph of order 5 which is Eulerian but not Hamiltonian. Motivate your answer. [4]

4. (a) Formulate the 5-colour theorem. [2]

(b) Here is an incomplete proof of the 5-colour theorem. Complete the proof by providing the missing part indicated by '...'. [6]

Let G be any connected plane graph. We proceed by induction on the number n of vertices. If $n \leq 5$ the graph is obviously 5-colourable. So assume that all planar graphs with fewer than n vertices, $n > 5$, are 5-colourable, and suppose that G is a plane graph with n vertices. By a previous theorem there is a vertex v in G such that deg $v \leq 5$. We delete v from G to obtain the plane graph $G - v$. By the induction hypothesis $G - v$ is 5-colourable. Let a 5-colouring of $G - v$ with colours 1,2,3,4, and 5 be given. If some colour is not used to colour the vertices adjacent to v in G then that colour may be used to colour v producing a 5-colouring of G. If not, v has degree 5 and all 5 colours are used to colour the 5 vertices adjacent to v.
Assume, without loss of generality, that the vertices adjacent to v in G are v_1, v_2, v_3, v_4 and v_5, and that v_i is coloured with colour i, $1 \leq i \leq 5$. We may also assume that v_1, v_2, v_3, v_4 and v_5 are arranged cyclically around v, in that order.
Now consider the vertices v_1 and v_3 and let H be the subgraph of $G - v$ induced by those vertices coloured 1 or 3. If v_1 and v_3 belong to different components of H, then we can interchange the colours (1 and 5) assigned to the vertices in the component of H containing v_1 and still have a 5-colouring of $G - v$. In the 5-colouring obtained in this way none of the vertices v_1, v_2, v_3, v_4 or v_5 is coloured 1. So we may colour v with colour 1, producing the desired 5-colouring of G.
So now suppose that v_1 and v_3 belong to the same component of H. ...

(c) Is the converse of the 5-colour theorem true? Motivate your answer by providing either a short proof or a counter-example. [3]