UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE
FAKULTEIT NATUURWETENSKAPPE

DEPARTMENT OF PURE AND APPLIED MATHEMATICS

MODULE **MAT2A10**
Sequences, Series and Vector Calculus

CAMPUS **APK**

EXAM **JUNE 2016**

EXAMINER	C DUNCAN
INTERNAL MODERATOR	F SCHULZ
DURATION	2 HOURS
MARKS	50

SURNAME AND INITIALS ____________________

STUDENT NUMBER ____________________

CONTACT NUMBER ____________________

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INSTRUCTIONS:
1. ANSWER ALL QUESTIONS ON THE PAPER IN PEN
2. CALCULATORS ARE ALLOWED
3. INDICATE **CLEARLY** ANY ADDITIONAL WORKING OUT

Question 1 [5]

For questions (1.1) - (1.5), please circle only **ONE** correct answer:

(1.1) Let $\sum a_n$ be a series of real numbers. What does it mean for this series to be convergent?

(a) The series $\sum a_n$ will always and only converge to 0.

(b) The sequence $\{a_n\}$ is always monotonic and bounded.

(c) The sequence of partial sums $\{s_n\}$ of $\sum a_n$ always and only converges to 0.

(d) There is a real number L such that the sequence $\{a_n\}$ converges to L.

(e) There is a real number L such that the sequence of partial sums $\{s_n\}$ of $\sum a_n$ converges to L.

(1.2) Suppose $\sum a_n$ is a series of real numbers with $\lim_{n\to\infty} a_n = 0$. Then:

(a) The series $\sum a_n$ is convergent.

(b) The series $\sum a_n$ is divergent.

(c) No conclusion can be drawn about the convergence or divergence of the series.

(1.3) Which of the following statements are true:

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ for a series $\sum a_n$, then the series is divergent.

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ for a series $\sum a_n$, then the series is convergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$ for a series $\sum a_n$, then the series is convergent.

(iv) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$ for a series $\sum a_n$, then the convergence of the series is inconclusive.

(a) i & ii (b) i, iii, & iv (c) ii & iv (d) i & iv

(1.4) The Taylor series of e^x about the point $x = -1$ is given by:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{e^n(n!)}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(x+1)^n}{e(n!)}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(x+1)^n}{(e^2\, n)!}$

(1.5) Let C be a smooth curve defined by a vector function $\mathbf{r}$ with unit tangent vector $\mathbf{T}$, binormal vector $\mathbf{B}$ and normal vector $\mathbf{N}$. Then the following statements are true:

(i) $\mathbf{T} \perp \mathbf{T}'$

(ii) $\mathbf{T} \perp \mathbf{B}$

(iii) $\mathbf{B} \perp \mathbf{N}$

(a) i & ii (b) ii & iii (c) i, ii & iii

Question 2 [7]

Using an appropriate method, determine if the following series are convergent or divergent:

(2.1) $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n\sqrt{\ln(n)+1}}$ (3)

(2.2) $\displaystyle\sum_{k=1}^{\infty} \frac{k^2\, 2^{k-1}}{(-5)^k}$ (2)

(2.3) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{1 + n^{-1}}{n}$ (2)

Question 3 [4]

Prove the Monotonic Sequence Theorem.

Question 4 [5]

(4.1) State the Ratio Test for series. (3)

(4.2) Hence, or otherwise, determine whether the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$$

(2)

Question 5 [6]

(5.1) Prove that $\cos x$ is equal to the sum of its Maclaurin series. (3)

(5.2) Use power series expansion to evaluate the following limit

$$\lim_{x \to 0} \frac{\cos(x^2) - 1}{x^4}$$

(3)

Question 6 [3]

Find the sum of the following series

$$-\frac{\pi}{6 \cdot 1!} + \frac{\pi^3}{6^3 \cdot 3!} - \frac{\pi^5}{6^5 \cdot 5!} + \frac{\pi^7}{6^7 \cdot 7!} - \dots$$

Question 7 [4]

Let $\mathbf{r}(t) = \left\langle \sqrt{t}\,,\ \dfrac{\ln t}{t^2 - 1}\,,\ 2t^2 \right\rangle$.

(7.1) Determine $\lim\limits_{t \to 1} \mathbf{r}(t)$. (2)

(7.2) Is $\mathbf{r}(t)$ continuous at $t = 1$? Motivate your answer clearly. (2)

Question 8 [3]

The helix $\mathbf{r}_1(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + t\ \mathbf{k}$ intersects the curve $\mathbf{r}_2(t) = (1+t)\ \mathbf{i} + t^2\ \mathbf{j} + t^3\ \mathbf{k}$ at the point $(1, 0, 0)$. Find the angle of intersection of these curves.

Question 9

[4]

Prove that the curvature of the curve given by the vector function $\mathbf{r}(t)$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Question 10 [5]

Consider a curve whose position is given by the vector function

$$\mathbf{r}(t) = e^t \cos t \ \mathbf{i} + \ \mathbf{j} + e^t \sin t \ \mathbf{k}$$

(10.1) Reparametrise the above curve with respect to arc length measured from the point $(1, 1, 0)$ in the direction of increasing t. (4)

(10.2) Determine at what position we are on the curve after we have traveled a distance of $\sqrt{2}$ units. (1)

Question 11 [5]

Let $\mathbf{v}$ be the velocity, v the speed and $\mathbf{a}$ the acceleration of a particle whose position is given by the vector function $\mathbf{r}$.

(11.1) Prove that $\mathbf{a} = v'\,\mathbf{T} + \kappa v^2\,\mathbf{N}$ (3)

(11.2) Let C be the curve given by the position vector $\mathbf{r}(t) = \langle 3t, -t, t^2 \rangle$. Determine the tangential component of the acceleration of a particle moving along the curve C. (2)