FACULTY OF SCIENCE
UNIVERSITY OF JOHANNESBURG

DEPARTMENT OF PURE AND APPLIED MATHEMATICS

MODULE	**MAT0CB2** ENGINEERING MULTIVARIABLE AND VECTOR CALCULUS
CAMPUS	**APK**
EXAM	**NOVEMBER 2016**

EXAMINER(S)	MRS C DUNCAN DR U KOUMBA
INTERNAL MODERATOR	MRS C MARAIS
DURATION	2 HOURS
MARKS	50

SURNAME AND INITIALS ______________________________

STUDENT NUMBER ______________________________

CONTACT NUMBER ______________________________

NUMBER OF PAGES: 1 + 14

INSTRUCTIONS:
1. ANSWER ALL QUESTIONS ON THE PAPER IN PEN
2. CALCULATORS ARE ALLOWED
3. INDICATE **CLEARLY** ANY ADDITIONAL PAGES USED

Question 1 [8]

For questions (1.1) - (1.8), please circle only **ONE** correct answer:

(1.1) Determine the range of the function

$$f(x, y, z) = \frac{3}{\sqrt{1 - x^2 - y^2 - z^2}}$$

(a) $(0, 3]$

(b) $(0, 1]$

(c) $\left[\frac{1}{3}, \infty\right)$

(d) $[3, \infty)$

(e) $\left[\frac{1}{3}, 3\right]$

(1.2) If $\mathrm{D}_{\mathbf{u}} f(0, 0) = c$, for any unit vector $\mathbf{u}$, then $c = 0$.

(a) True

(b) False

(1.3) Suppose $(1, 1)$ is a critical point of a function f with continuous second derivatives. In the case of $f_{xx}(1, 1) = 7$, $f_{xy}(1, 1) = 8$, $f_{yy}(1, 1) = 10$ what can you say about f?

(a) f has a local maximum at $(1, 1)$

(b) f has a saddle point at $(1, 1)$

(c) f has a local minimum at $(1, 1)$

(1.4) Find the volume of the solid that lies inside the cylinder $x^2 + y^2 = 9$ and the ellipsoid $2x^2 + 2y^2 + z^2 = 36$.

(a) 260.31

(b) 301.74

(c) 261.29

(d) 292.45

(e) 284.22

(1.5) Find the Jacobian of the transformation $x = 5\alpha \sin \beta$ and $y = 4\alpha \cos \beta$.

(a) 9α

(b) $-20\alpha \sin \beta \cos \beta$

(c) -20α

(d) $-\alpha$

(e) 36α

(1.6) Use Green's Theorem to evaluate $\int_C x^2 y \, dx - xy^2 \, dy$, where C is the circle $x^2 + y^2 = 4$ with counterclockwise orientation.

(a) -8π

(b) -4π

(c) 6π

(d) 2π

(e) none of these

(1.7) Given the position vector $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, with $|\mathbf{r}| = r$. Then, $\nabla \cdot (9r\mathbf{r}) = 36r$.

(a) True

(b) False

(1.8) The vector field $\mathbf{F} = xyz\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ can always be written as the curl of another vector field.

(a) True

(b) False

Question 2 [10]

(2.1) Prove that if f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

(4)

(2.2) Given the unit vector $\mathbf{u} = \langle a, b \rangle$ and that f has continuous second partial derivatives, use Clairaut's Theorem to show that

$$D_{\mathbf{u}}^2 f(x, y) = f_{xx}(x, y)\, a^2 + 2 f_{xy}(x, y)\, ab + f_{yy}(x, y)\, b^2$$

[Note that $D_{\mathbf{u}}^2 f = D_{\mathbf{u}}(D_{\mathbf{u}} f)$] (3)

(2.3) If $f(x, y) = xe^{2y}$, then find $D^2_{\mathbf{u}} f(0, 0)$ in the direction of $\mathbf{v} = \langle 4, 6 \rangle$. (4)

Question 3 [7]

Given $V = \int_0^1 \int_0^{1-x^2} \int_0^{1-x} dydzdx.$

(3.1) Sketch the region of integration. (3)

(3.2) Rewrite V in the order $dxdydz$. (4)

Question 4 [3]

Use appropriate coordinates to set up the integral $\iiint\limits_E y \, dV$, where E is the region that lies below the plane $z = x + 2$, above the xy-plane and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Question 5 [6]

Consider the double integral

$$\iint_{\Omega} (x+y)\, dx dy$$

with Ω the region bounded by $x = y,\ \ x = y + \pi,\ \ x = -2y$ and $x = -2y + \frac{\pi}{2}$.

(5.1) Determine the transformation $T(u, v) = (x, y)$ such that $x = g(u, v)$ and $y = h(u, v)$, and calculate the associated Jacobian. (3)

(5.2) Using (5.1), solve the integral $\iint_{\Omega} (x + y)\, dxdy$. (3)

Question 6 [6]

Consider the vector field $\mathbf{F} = \langle 3x^2 + y, 3xy^2 \rangle$.

(6.1) Determine whether $\mathbf{F}$ is conservative. (2)

(6.2) Consequently, determine the work done by $\mathbf{F}$ along the closed path C, with counter-clockwise orientation that encloses the region bounded by the graphs of $y = \sqrt{x}$, $y = 0$ and $x = 4$. (4)

Question 7 [4]

Prove that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

Question 8 [3]

Determine whether the vector field $\mathbf{F}$ is incompressible at the point $(0, 0, 3)$ if given

$$\mathbf{F} = e^x \sin y\,\mathbf{i} - e^x \cos y\,\mathbf{j} + \ln z\,\mathbf{k}$$

Question 9 [3]

If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then show that

$$\text{div curl } \mathbf{F} = 0.$$