UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE
FAKULTEIT NATUURWETENSKAPPE

DEPARTMENT OF PURE AND APPLIED MATHEMATICS

MODULE	**MAT0CA2** Engineering Sequences, Series and Vector Calculus
CAMPUS	**APK**
EXAM	**JUNE 2016**

EXAMINER(S)	C DUNCAN U KOUMBA
INTERNAL MODERATOR	C MARAIS
DURATION	2 HOURS
MARKS	50

SURNAME AND INITIALS ______________________________

STUDENT NUMBER ______________________________

CONTACT NUMBER ______________________________

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INSTRUCTIONS:
1. ANSWER ALL QUESTIONS ON THE PAPER IN PEN
2. CALCULATORS ARE ALLOWED
3. INDICATE **CLEARLY** ANY ADDITIONAL WORKING OUT

Question 1 [7]

Test the convergence or divergence of the following series:

(1.1) $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^{3/2}}$ (3)

(1.2) $\displaystyle\sum_{k=1}^{\infty} \frac{k^2\, 2^{k-1}}{(-5)^k}$ (2)

(1.3) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{1+n^{-1}}{n}$ (2)

Question 2 [7]

(2.1) State and prove the Direct Comparison Test. (5)

(2.2) Determine the convergence or divergence of the series $\sum_{n=1}^{\infty}(n^3+1)^{-1/2}$. (2)

Question 3 [11]

Your lecturer wants you to determine the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. Answer the following questions pertaining to this sum:

(3.1) Verify that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent. (2)

(3.2) Give the Maclaurin series for $f(x) = \frac{1}{1+x}$. (1)

(3.3) Deduce an expansion of $g(x) = \dfrac{1}{1+e^{-x}}$ in powers of e^{-x}. (2)

(3.4) Given that $g(x) = e^{x}\, h(x)$, prove that $\displaystyle\int_0^\infty h(x)\; dx \;=\; \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

[Hint: Use part (3.3)] (3)

(3.5) Using only techniques of integration, evaluate $\displaystyle\int_0^{\infty} \frac{e^{-x}}{1+e^{-x}}\, dx$. (2)

(3.6) Hence, provide the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. (1)

Question 4 [3]

Use power series expansion to evaluate the following limit

$$\lim_{x \to 0} \frac{\cos(x^2) - 1}{x^4}$$

Question 5 [2]

Find the sum of the following series

$$3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \dots$$

Question 6 [4]

Let $\mathbf{r}(t) = \left\langle \sqrt{t}\,,\ \dfrac{\ln t}{t^2 - 1}\,,\ 2t^2 \right\rangle$.

(6.1) Determine $\lim_{t \to 1} \mathbf{r}(t)$. (2)

(6.2) Is $\mathbf{r}(t)$ continuous at $t = 1$? Motivate your answer clearly. (2)

Question 7 [3]

The helix $\mathbf{r}_1(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + t\ \mathbf{k}$ intersects the curve $\mathbf{r}_2(t) = (1+t)\ \mathbf{i} + t^2\ \mathbf{j} + t^3\ \mathbf{k}$ at the point $(1, 0, 0)$. Find the angle of intersection of these curves.

Question 8

[4]

Prove that the curvature of the curve given by the vector function $\mathbf{r}(t)$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t)\ \times\ \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Question 9

[5]

Consider a curve whose position is given by the vector function

$$\mathbf{r}(t) = e^t \cos t \ \mathbf{i} + \ \mathbf{j} + e^t \sin t \ \mathbf{k}$$

(9.1) Reparametrise the above curve with respect to arc length measured from the point $(1, 1, 0)$ in the direction of increasing t. (4)

(9.2) Determine at what position we are on the curve after we have traveled a distance of $\sqrt{2}$ units. (1)

Question 10 [5]

Let $\mathbf{v}$ be the velocity, υ the speed and $\mathbf{a}$ the acceleration of a particle whose position is given by the vector function $\mathbf{r}$.

(10.1) Prove that $\mathbf{a} = \upsilon' \, \mathbf{T} + \kappa \upsilon^2 \, \mathbf{N}$ (3)

(10.2) Let C be the curve given by the position vector $\mathbf{r}(t) = \langle 3t, -t, t^2 \rangle$. Determine the tangential component of the acceleration of a particle moving along the curve C. (2)