UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE

DEPARTMENT OF PURE AND APPLIED MATHEMATICS

MODULE	**ASMA2A1** SEQUENCES, SERIES AND VECTOR CALCULUS
CAMPUS	**APK**
EXAM	**NOVEMBER 2016**

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DURATION	2.5 HOURS
MARKS	50

SURNAME AND INITIALS ______________________________

STUDENT NUMBER ______________________________

IDENTITY NUMBER ______________________________

NUMBER OF PAGES: 1 + 13

INSTRUCTIONS:
1. ANSWER ALL QUESTIONS ON THE PAPER IN PEN
2. CALCULATORS ARE ALLOWED
3. INDICATE **CLEARLY** ANY ADDITIONAL WORKING OUT

Question 1

Use the precise definition of a limit of a sequence to show that [3]

$$\lim_{n\to\infty} \frac{3n^2+1}{2n^2+n} = \frac{3}{2}.$$

Question 2

State and prove the Alternating Series Test. [6]

Question 3

Test the following series for convergence or divergence: [8]

(3.1) $\sum_{n=1}^{\infty}(-1)^n\left(\arctan n - \frac{\pi}{2}\right)$ (4)

(3.2) $\sum_{n=1}^{\infty} b^{\ln n}$, where $b \in \mathbb{R}$ satisfies $0 < b < e^{-1}$. **Hint:** Show that the series is a p-series. (4)

Question 4

Prove or disprove: If $\sum a_n$ and $\sum b_n$ are convergent series, then $\sum a_n b_n$ is a convergent series. [3]

Question 5 [7]

(5.1) Use Taylor's Inequality and the fact that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all $x \in \mathbb{R}$ to show that (3)

$$e^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} \quad \text{for all } \; x \in \mathbb{R}.$$

(5.2) By using the result in (5.1), find a Maclaurin series expansion for $f(x) = \cosh(2x)$. (2)

(5.3) Why can we conclude from (5.2) that $f(x) \geq 1 + 2x^2 + \frac{2}{3}x^4$ for all $x \in \mathbb{R}$? (2)

Question 6

Consider the approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

For what values of x is this approximation accurate to within 10^{-4}? [2]

Question 7

For questions (7.1) - (7.5), please circle only **ONE** correct answer: [5]

(7.1) Two particles travel along the space curves

$$\mathbf{r}_1(t) = \left\langle \cos t, \sin^2 t, t^2 - 4\pi^2 \right\rangle \text{ and } \mathbf{r}_2(t) = \left\langle 1+t, t^2, t^3 \right\rangle.$$

What can be said about the motion of the particles? (1)

(a) The particles collide.

(b) The paths of the particles intersect, but they do not collide.

(c) The paths of the particles do not intersect.

(d) None of the above.

(7.2) If $\mathbf{r}'(t) = \langle t^2, t\cos \pi t, \sin \pi t\rangle$ and $\mathbf{r}(0) = \left\langle 0, \frac{1}{\pi^2}, -\frac{1}{\pi} \right\rangle$, then (1)

(a) $\mathbf{r}(t) = \left\langle \frac{t^3}{3}, \frac{t}{\pi}\sin \pi t + \frac{1}{\pi^2}\cos \pi t, -\frac{1}{\pi}\cos \pi t \right\rangle$.

(b) $\mathbf{r}(t) = \left\langle \frac{t^3}{3}, \frac{t}{\pi}\cos \pi t - \frac{1}{\pi^2}\sin \pi t, -\frac{1}{\pi}\cos \pi t \right\rangle$.

(c) $\mathbf{r}(t) = \left\langle \frac{t^3}{3}, -\frac{t}{\pi}\sin \pi t - \frac{1}{\pi^2}\cos \pi t, \frac{1}{\pi}\cos \pi t \right\rangle$.

(d) $\mathbf{r}(t) = \left\langle \frac{t^3}{3}, \frac{t}{\pi}\sin \pi t - \frac{1}{\pi^2}\cos \pi t, \frac{1}{\pi}\cos \pi t \right\rangle$.

(7.3) If $\mathbf{r}(t)$ is a differentiable vector function, then (1)

$$\frac{d}{dt}\left|\mathbf{r}(t)\right| = \left|\mathbf{r}'(t)\right|.$$

(a) True (b) False

(7.4) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, and that f is a differentiable real-valued function. Consider the following differentiation formulas:

(i) $\frac{d}{dt}\left[\mathbf{u}(t)\cdot\mathbf{v}(t)\right] = \mathbf{v}'(t)\cdot\mathbf{u}(t) + \mathbf{u}'(t)\cdot\mathbf{v}(t)$.

(ii) $\frac{d}{dt}\left[\mathbf{u}(t)\times\mathbf{v}(t)\right] = \mathbf{v}'(t)\times\mathbf{u}(t) + \mathbf{u}'(t)\times\mathbf{v}(t)$.

(iii) $\frac{d}{dt}\left[\mathbf{u}\left(f(t)\right)\right] = f'(t)\mathbf{u}'\left(f(t)\right)$.

Which of the above statements are always true? (1)

(a) i, ii, iii (b) i, ii (c) iii (d) i, iii (e) ii, iii

(7.5) The curvature of $y = e^x$ at $x = 0$ is (1)

(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{2}$ (d) $\frac{1}{2\sqrt{3}}$ (e) 2

Question 8

Find the length of the curve

$$\mathbf{r}(t) = \left\langle \frac{t^3}{3}, \frac{t^2}{\sqrt{2}}, t \right\rangle$$

from $t = 0$ to $t = 3$. [4]

Question 9

Show that if there is a $c \in \mathbb{R}$ such that $|\mathbf{r}(t)| = c$ for all t, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$. [3]

Question 10

Prove that the curvature of a curve C with vector function $\mathbf{r}(t)$ is given by the following formula:[4]

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

Question 11

Find the tangential and normal components of the acceleration vector of a particle with position function $\mathbf{r}(t) = \langle t, 2t, t^2 \rangle$. [5]