UNIVERSITY OF JOHANNESBURG

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FACULTY OF SCIENCE

DEPARTMENT OF APPLIED MATHEMATICS

MODULE: **APM1B10**
INTRODUCTION TO DYNAMICS
CAMPUS: **APK**

DECMBER EXAMINATION

DATE: 01/12/2016 **SESSION: 12:30 - 15:30**

ASSESSORS: **DR JSC PRENTICE**
DR M MOLELEKOA

INTERNAL MODERATOR: **PROF C M VILLET**

DURATION: **2 HOURS** **MARKS:** **50**

NUMBER OF PAGES: 5 PAGES

INSTRUCTIONS:

SYMBOLS HAVE THEIR USUAL MEANING.
PHYSICAL QUANTITIES ARE IN SI UNITS AND ANGLES ARE IN RADIANS.
CALCULATORS ARE PERMITTED.
WORK TO A PRECISION OF AT LEAST THREE DECIMAL PLACES IN ALL NUMERICAL CALCULATIONS.
ANSWER EACH OF THE FIRST THREE THE QUESTIONS.
ANSWER EITHER QUESTION FOUR OR QUESTION FIVE, NOT BOTH.

QUESTION 1

(a) Show that

$$\widehat{r}\cdot\widehat{n}=\frac{v_\theta a_n}{v_\theta a_r - v_r a_\theta}.$$

(b) The equation of the trajectory of a particle is given by

$$\mathbf{r}\,(t)=3t^4\widehat{x}-\sin\,(2t)\,\widehat{y}\equiv\left(3t^4,-\sin\,(2t)\right).$$

Find, for $t=\frac{1}{2}$,

(i) $\mathbf{v}$ and $\mathbf{a}$,

(ii) $\widehat{\tau}$ and $\widehat{n}$,

(iii) the tangential and normal components of $\mathbf{a}$,

(iv) the radius of curvature ρ, and the centre of curvature $\mathbf{C}=\mathbf{r}+\rho\widehat{n}$.

[12]

QUESTION 2

An object of mass $m = 1$ kg, moving horizontally, hits a windscreen travelling in the opposite direction. The windscreen is tilted at an angle $\theta=\frac{\pi}{4}$ rad to the horizontal. A force $\mathbf{F}$ acts on the object, for a duration of T seconds, such that $\mathbf{F}$ is constant in magnitude and is directed normal (perpendicular) to the windscreen. At impact, the object has speed v_i and the windscreen has speed $v_w = 2v_i$. Impose a frame of reference in which the windscreen is stationary, and in which gravity acts in the usual downward direction. Show that, if the object bounces vertically upwards off the windscreen at 2 ms^{-1}, then

$$T=\frac{3v_i-2}{g},$$

where g is the gravitational acceleration.

[13]

QUESTION 3

Calculate the directional derivative $\frac{\partial\psi}{\partial s}$ of

$$\psi = -xz + yz^2$$

in the direction

$$3\widehat{x} - 2\widehat{y} - 1\widehat{z}$$

at the point

$$(4, 1, -2)\,.$$

(a) Use the parameterization $\mathbf{r} = \mathbf{r}_0 + s\widehat{s}$.

(b) Use $\nabla\psi$.

[12]

THE PREVIOUS THREE QUESTIONS WERE COMPULSORY.
MAKE SURE THAT YOU HAVE ANSWERED ALL OF THEM.

QUESTIONS FOUR AND FIVE ARE ON THE NEXT PAGE.
ANSWER EITHER ONE OF THEM, NOT BOTH.

QUESTION 4

Calculate $\int_\Gamma \mathbf{a} \cdot d\mathbf{r}$ with $\mathbf{a} = \left(z - 2y,\ 2xy^2,\ 3x - z^2\right)$, and where the path Γ is defined by

$$\begin{aligned} y &= x^2 + 1 \\ z &= x^2 - 2y. \end{aligned}$$

In the integral, the lower limit is $(0, 1, -2)$, and the upper limit is $(1, 2, -3)$. You should find that the value of the integral is $\frac{298}{105}$.

[13]

OR

QUESTION 5

The coefficient of friction between the tyres of a braking car and the road is μ. At the moment the brakes are applied, the car has speed v_0. Assume that the road is inclined upwards at an angle θ, and let d denote the distance in which the car comes to a stop. Using the relationship

$$W = \int_\Gamma \mathbf{F} \cdot d\mathbf{r} = \Delta T,$$

show that, if $\theta = \frac{\pi}{4}$ rad,

$$v_0 = \sqrt{\sqrt{2} g d \,(1 + \mu)},$$

where g is the gravitational acceleration. In this problem, the friction is **not** to be treated as a conservative force, so that the principle of conservation of energy is **not** applicable.

[13]

INFORMATION

$$\begin{aligned}
\mathbf{r} &= (x,y) = (r\cos\theta, r\sin\theta) \\
\mathbf{v} &= v_r\widehat{r} + v_\theta\widehat{\theta} = v_\tau\widehat{\tau} + v_n\widehat{n} \\
&= \dot{r}\widehat{r} + r\dot{\theta}\widehat{\theta} = v\widehat{\tau} \\
\mathbf{a} &= a_r\widehat{r} + a_\theta\widehat{\theta} = a_\tau\widehat{\tau} + a_n\widehat{n} \\
&= \left(\ddot{r} - r\dot{\theta}^2\right)\widehat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\widehat{\theta} = \dot{v}\widehat{\tau} + \left(\frac{v^2}{\rho}\right)\widehat{n} \\
\widehat{r} &= (\cos\theta, \sin\theta) \qquad \widehat{\theta} = (-\sin\theta, \cos\theta) \\
\widehat{\tau} &= (\cos\psi, \sin\psi) \qquad \widehat{n} = (-\sin\psi, \cos\psi)
\end{aligned}$$

$$\begin{aligned}
\mathbf{I} &= \int_{t_i}^{t_f} \mathbf{F} dt = m\mathbf{v}_f - m\mathbf{v}_i \\
\ddot{x} &= -\omega^2 x \\
\Rightarrow x &= x_0 \sin(\omega t + B)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\Psi}{\partial s} &= \nabla\Psi\cdot\widehat{s} \\
d\mathbf{r} &= (dx, dy) \text{ in two dimensions} \\
d\mathbf{r} &= (dx, dy, dz) \text{ in three dimensions}
\end{aligned}$$

$$\begin{aligned}
T = \frac{m(\mathbf{v}\cdot\mathbf{v})}{2} &= \frac{mv^2}{2} \\
W = \int_\Gamma \mathbf{F}\cdot d\mathbf{r} &= \Delta T \\
\text{Newtonian gravity:}\quad \mathbf{F} &= -\frac{GMm}{r^2}\widehat{r}
\end{aligned}$$