UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE

DEPARTMENT OF PURE AND APPLIED MATHEMATICS

MODULE	**MAT2B20** LINEAR ALGEBRA B
CAMPUS	**APK**
EXAM	**NOVEMBER EXAM**

DATE 13/11/2014

SESSION 12:30 – 15:30

ASSESSORS DR G BRAATVEDT, MS C MARAIS

INTERNAL MODERATOR MR G VAN DRIMMELEN

DURATION 2 HOURS

MARKS 60

SURNAME AND INITIALS: ______________________________

STUDENT NUMBER: ______________________________

CONTACT NR: ______________________________

NUMBER OF PAGES: 1 + 12 PAGES

INSTRUCTIONS:

1. ANSWER ALL THE QUESTIONS ON THE PAPER IN PEN.
2. NO CALCULATORS ARE ALLOWED.
3. SHOW ALL CALCULATIONS AND MOTIVATE ALL ANSWERS.
4. GOOD LUCK – WE HOPE YOU WRITE WELL :)

Question 1 [6]

Let $A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$.

(a) Determine the eigenvalues of A. (3)

(b) Determine which of the following are eigenvectors of A:

$$(0, 2, 0), (3, 1, 1), (-1, 3, 1), (1, 0, 1)$$

(2)

(c) Hence, set up a matrix product representing A^{13} (you need not simplify). (1)

Question 2 [4]

Consider P_1 equipped with the evaluation inner product:

$$\langle \overline{p}, \overline{q} \rangle = p(-1)q(-1) + p(1)q(1)$$

for $\overline{p}, \overline{q} \in P_1$.

(a) Determine $d(1+2x, x)$. (2)

(b) For which value(s) of s and t will the inner product

$$\langle \overline{p}, \overline{q} \rangle = p(s)q(s) + p(t)q(t)$$

coincide with the inner product

$$\langle \overline{p}, \overline{q} \rangle = a_0 b_0 + a_1 b_1$$

where $p(x) = a_0 + a_1 x$, $q(x) = b_0 + b_1 x$. (2)

Question 3 [6]

(a) Prove that if W is a subspace of an inner product space V, then $W^{\perp}$ is a subspace of V. (3)

(b) State the Projection Theorem. (2)

(c) Give a condition on a matrix A under which it will have a QR factorization. (1)

Question 4 [5]

Let $W = \text{span}((1,1,1),(0,1,0))$, and $\overline{b} = (1,2,2)$.

(a) Determine $\text{proj}_{W^\perp}\overline{b}$. (3)

(b) Hence, determine a least squares solution to the system: (2)

$$\begin{aligned} x \quad &= 1 \\ x + y &= 2 \\ x \quad &= 2 \end{aligned}$$

Question 5 [8]

Suppose that $A = \begin{bmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{bmatrix}$ and the characteristic polynomial for A is $-\lambda(\lambda - 9)^2$.

(a) Determine an orthogonal basis for $\mathbb{R}^3$ consisting of eigenvectors of A. (4)

(b) Hence, calculate P and D that orthogonally diagonalize A. (2)

(c) Determine a symmetric matrix B such that $B^2 = A$. (2)

Question 6 [4]

Consider the quadratic form $3x_1^2 + 5x_2^2 + 6x_3^2 + 4x_4^2 - 2x_1x_3 - 2x_2x_4 - x_3x_4$.

(a) Write the quadratic form in matrix form $\overline{x}^T A \overline{x}$. (1)

(b) Determine whether the quadratic form is positive definite, negative definite, or indefinite. (2)

(c) Hence, is the A obtained in (a) invertible? Explain. (1)

Question 7 [6]

Let $T : P_2 \to M_{22}$ be the linear transformation defined by

$$T(\overline{p}) = \begin{bmatrix} p(1) - p(0) & p(2) - p(0) \\ p(-1) - p(0) & p(-2) - p(0) \end{bmatrix}.$$

(a) Let $B = \{1, x, x^2\}$ and $C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. Determine the matrix representation of T relative to B and C. (4)

(b) Hence, calculate $[T(1 + 2x + x^2)]_C$. (2)

Question 8 [6]

(a) Prove that if A is orthogonally diagonalizable, then A is symmetric. (2)

(b) Let $T : V \to W$ be a linear transformation.
Prove that T is one-to-one if and only if $\ker(T) = \{\bar{0}\}$. (4)

Question 9 [10]

Let $T : M_{22} \to P_2$ be the transformation defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b) + (b-c)x + (d-c)x^2$$

(a) Show that T is linear. (3)

(b) If $S : P_2 \to \mathbb{R}^2$ is defined by

$$S(a + bx + cx^2) = (b, c),$$

determine $S \circ T$ explicitly. (2)

Question 10

Let V be the space of skew-symmetric [$A = -A^T$] 3×3 matrices. Identify a subspace of $\mathbb{R}^\infty$ [the space of all real-valued sequences] to which V is isomorphic. [2]

Question 11

Let A be an $n \times n$ orthogonal matrix. Show that the eigenvalues of A are either 1 or -1. [3]