UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE

APM0137
APPLIED MATHEMATICS HONOURS EXAM
NUMERICAL ANALYSIS B
2014

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TIME 3 hrs
MARKS 80

1. This paper consists of four questions.
2. Answer all four questions.
3. This paper consists of five pages.

QUESTION 1

a) Determine the stability function $R(h\lambda)$ for the Runge-Kutta method

$$\begin{array}{c|ccc} \frac{1}{2} & \frac{1}{2} & & \\ \frac{3}{4} & 0 & \frac{3}{4} & \\ \hline & \frac{2}{9} & \frac{3}{9} & \frac{4}{9} \end{array}.$$

b) Show that the parabolic problem

$$\begin{aligned} u_t &= \alpha^2 u_{xx} \\ x &\in (0, L), \qquad t > 0 \\ u(0,t) &= u(L,t) = 0, \qquad u(x,0) = g(x) \end{aligned}$$

can be solved approximately by means of a system of ODEs, which has Jacobian J given by

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial w_1} & \frac{\partial f_1}{\partial w_2} & \cdots & \frac{\partial f_1}{\partial w_{n-1}} & \frac{\partial f_1}{\partial w_n} \\ \frac{\partial f_2}{\partial w_1} & \frac{\partial f_2}{\partial w_2} & & \frac{\partial f_2}{\partial w_{n-1}} & \frac{\partial f_2}{\partial w_n} \\ \vdots & & \ddots & & \vdots \\ \frac{\partial f_{n-1}}{\partial w_1} & \frac{\partial f_{n-1}}{\partial w_2} & & \frac{\partial f_{n-1}}{\partial w_{n-1}} & \frac{\partial f_{n-1}}{\partial w_n} \\ \frac{\partial f_n}{\partial w_1} & \frac{\partial f_n}{\partial w_2} & \cdots & \frac{\partial f_n}{\partial w_{n-1}} & \frac{\partial f_n}{\partial w_n} \end{bmatrix} = \frac{\alpha^2}{h^2} \begin{bmatrix} -2 & 1 & 0 & & & & & \\ 1 & -2 & 1 & 0 & & & & \\ 0 & 1 & -2 & 1 & 0 & & & \\ & & & \ddots & & & & \\ & & 0 & 1 & -2 & 1 & 0 \\ & & & 0 & 1 & -2 & 1 \\ & & & & 0 & 1 & -2 \end{bmatrix},$$

where h is an appropriate discretization parameter.

c) Assuming that the system in (b) is to be solved using Euler's method, derive a condition for stability in the numerical solution by imposing a suitable condition on the Jacobian of the system.

[20]

HINT: Eigenvalues of an $N \times N$ tridiagonal matrix $\{b, a, c\}$

$$\phi_j = a + 2\sqrt{bc}\cos\left(\frac{j\pi}{N+1}\right) \qquad j = 1, 2, ..., N$$

QUESTION 2

a) Set up a linear system whose solution approximates the solution of

$$\nabla^2 u\,(x,y) = 2y \tag{1}$$

on

$$A = \{(x,y)\,|0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 2\}$$

subject to the boundary conditions

$$\begin{aligned} u\,(x,0) &= x \\ \left.\frac{\partial u}{\partial y}\right|_{(x,2)} &= x^2 \\ u\,(1,y) &= 1+y \\ \left.\frac{\partial u}{\partial x}\right|_{(0,y)} &= 1. \end{aligned}$$

Use the discretization

$$x_i = \frac{i}{4} \qquad y_j = \frac{j}{2}$$

for $i, j = 0, 1, \ldots, 4$. Make use of second-order approximations to the derivatives in the PDE and in the boundary conditions.

b) If this linear system is solved, we find that the approximate solution at each $(x,y) \in A$ is, in fact, equal to the exact solution evaluated at each $(x,y) \in A$. Explain this given that the analytical solution of (1) is

$$u\,(x,y) = x + x^2 y.$$

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QUESTION 3

(a) Assume that the method

$$\begin{array}{c|cc} 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

is used to control the local error in Euler's method, via local extrapolation, for the initial-value problem

$$y' = xy, \quad y(2) = y_0.$$

If a tolerance δ is placed on the local error, show that the appropriate stepsize on the first step is given by

$$h \leqslant \sqrt{\frac{2\delta}{5|y_0|}}.$$

You may assume $h \ll 1$.

(b) Show that, if $y_1 > y_2, y_2 \neq 0$, then the system

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -y_1 - \ln y_2 \\ \frac{y_1^2+1}{8} \end{bmatrix}$$

has at least one stiff eigenvalue.

[20]

QUESTION 4

Consider the equation

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = 0. \qquad (2)$$

a) Use the *von Neumann ansatz*

$$w_i^j = \xi^j e^{\mathbf{i}\theta i}$$

to show that the Crank-Nicolson method applied to (2) is unconditionally stable.

b) Show that the leading term of the *local truncation error* τ_i^{j+1} of Richardson's method

$$\frac{w_i^{j+1} - w_i^{j-1}}{2k} - a^2 \left(\frac{w_{i+1}^j - 2w_i^j + w_{i-1}^j}{h^2} \right) = 0$$

for (2), is

$$\frac{k^2}{3} \left. \frac{\partial^3 u}{\partial t^3} \right|_{(x_i,t_j)} - \frac{a^2 h^2}{6} \left. \frac{\partial^4 u}{\partial x^4} \right|_{(x_i,t_j)}.$$

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