UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE

Applied Mathematics
APK Campus
APM0117 DIFFERENTIAL EQUATIONS B
3 NOVEMBER 2014

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SECOND EXAMINER	PROF J MOITSHEKI
TIME 2 HOURS	**MARKS** 40

Please note the following:

1. Answer all the questions.
2. This paper consists of 3 pages including this one.

Question 1

Use the method of separation of variables to solve the initial-Neumann problem:

$$\begin{cases} u_t - u_{xx} = 0, & 0 < x < 1, \quad t > 0 \\ u(x,0) = x, & 0 < x < 1 \\ u_x(0,t) = u_x(1,t) = 0, & t > 0 \end{cases}$$

[10]

Question 2

Let $l > 0$ be a positive integer, $S = (0,\ l) \times (0,\ \infty)$ and suppose $u(x,t) \in C^{1,2}(\bar{S})$ is the solution of the diffusion problem

$$\begin{cases} u_t - u_{xx} = 0, & (x,t) \in S \\ u(x,0) = \dfrac{1}{l^2} x\,(l-x), & 0 \le x \le l \\ u_x(0,t) = u_x(l,t) = 0, & t \in (0,\infty) \end{cases}$$

Let $\|\cdot\|$ denote the spatial L^2 norm over $[0,l]$. Then, show that

(a) $\|u(\cdot,0)\| = \sqrt{\frac{l}{30}}$

(b) $\frac{d}{dt}\left(\|u(\cdot,t)\|^2\right) = -2\,\|u_x(\cdot,t)\|$

(c) $\|u(\cdot,t)\| \le \sqrt{l}\,\|u_x(\cdot,t)\|$

[10]

Question 3

Consider the wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & 0 < x < L, \quad t > 0 \\ u_x(0,t) = a(t),\ \ u_x(L,t) = b(t), & t \ge 0 \\ u(x,0) = \phi(x),\ \ u_t(x,0) = \psi(x) & 0 \le x \le L \end{cases}$$

By using energy methods, or otherwise, prove that the problem has at most one solution. [8]

Question 4

Define Green's function $G(x, x_0)$ over $\mathbb{R}^3$ so that

$$LG(x, x_0) = \delta(x - x_0),$$

where $\delta(x)$ is a delta function such that $\iiint \delta(x - x_0) f(x_0)\, dV = f(x)$ and L is the Laplacian operator. If $\phi(x),\ x \in \mathbb{R}^3$ is defined by

$$\phi(x) = \iiint G(x, x_0) f(x_0)\, dV,$$

prove that $\phi(x)$ is a solution Poisson's equation $L\phi = f$. [8]

Question 5

Classify the following PDE as elliptic, hyperbolic or parabolic:

$$y^2 u_{xx} - 2\, xy\, u_{xy} + x^2 u_{yy} = \frac{y^2}{x}\ u_x + \frac{x^2}{y}\ u_y.$$

[4]

********** END OF EXAM **********