UNIVERSITY OF JOHANNESBURG

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FACULTY OF SCIENCE

DEPARTMENT:	**PURE AND APPLIED MATHEMATICS**
MODULE:	**APM2B10** INTRODUCTION TO NUMERICAL ANALYSIS
CAMPUS:	**AUCKLAND PARK KINGSWAY**

SUPPLEMENTARY DECEMBER EXAMINATION

DATE: 03/12/2014 **SESSION: 08:30 – 11:30**

ASSESSOR **MR KD ANDERSON**
INTERNAL MODERATOR **DR JSC PRENTICE**

DURATION: 2 HOURS 30 MINUTES **MARKS: 50**

NUMBER OF PAGES 3 PAGES

INSTRUCTIONS

ANSWER ALL THE QUESTIONS.
SHOW ALL CALCULATIONS.
POCKET CALCULATORS MAY BE USED.
WORK TO A PRECISION OF AT LEAST THREE DECIMAL PLACES.
SYMBOLS HAVE THEIR USUAL MEANING.

QUESTION 1 [15 marks]

(a) Consider the following differential equation (5)

$$\frac{dy}{dt} = -5y + 5t^2 + 2t$$

with initial value $y(0) = \frac{1}{3}$. Approximate a solution to this differential equation on the interval $[0, 1]$ using the second-order Runge-Kutta method with step size $h = \frac{1}{2}$.

(b) Determine the upper bound on the magnitude of the global error of the differential equation in (a), given (10)

$$|\epsilon_{i+1}| = \left|\frac{h^3}{3!}y'''(\xi)\right|, \quad \xi \in (t_i, t_{i+1})$$

and

$$F(t, y) = \frac{10y - 10t^2 + 16t + 9}{8}$$

QUESTION 2 [10 marks]

(a) Compute the first two iterations using the linear interpolation method with initial points $(x_1, y_1) = (-2, 0.541)$ and $(x_2, y_2) = (0, -1)$ to approximate the root of $f(x) = e^x(x^2 - 1)$. (3)

(b) Derive (7)

$$y_i' = \frac{-y_{i+2} + 4y_{i+1} - 3y_i}{2h}.$$

What is the order of the error?

QUESTION 3 [15 marks]

(a) How many steps (N) and what step size (h) are required to integrate (10)

$$\int_{-2}^{0} x^2 e^x \, dx,$$

using the composite Trapezium rule, accurate to $\epsilon = 0.1$?

(b) Perform the integration using the information obtained in part (a). (5)

QUESTION 4 [10 marks]

Find the series expansion of

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

in terms of Chebyshev polynomials by using the orthogonality property of the Chebyshev polynomials.

Formulae

$$f(x) = \frac{1}{2}c_0T_0(x) + c_1T_1(x) + c_2T_2(x) + \cdots$$

$$\begin{aligned} c_k &= \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}}\, dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} f(\cos\theta)\cos(k\theta)\, d\theta \end{aligned}$$

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left(y_0 + y_N + 2\sum_{i=1}^{N-1} y_i\right)$$

$$|\Delta| \leq \frac{h^2\,(b-a)\,M}{12} \quad \text{where} \quad M = \max_{[a,b]} |f''\,(x)|$$

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left(y_0 + y_N + 4y_1 + \sum_{i=1}^{N-1}(2y_{2i} + 4y_{2i+1})\right)$$

$$|\Delta| \leq \frac{h^4\,(b-a)\,K}{180} \quad \text{where} \quad K = \max_{[a,b]} \left|f^{(4)}\,(x)\right|$$

$$y_{m+1} = y_m + hf(x_m, y_m)$$

$$\begin{aligned} k_1 &= hf(x_m, y_m) \\ k_2 &= hf(x_m + h, y_m + k_1) \\ y_{m+1} &= y_m + \tfrac{1}{2}(k_1 + k_2) \end{aligned}$$

$$\Delta_n = \sum_{j=1}^{n}\left[\left(\frac{1}{\alpha_n}\right)\left(\prod_{k=j}^{n}\alpha_k\right)\right]\varepsilon_j$$

$$\alpha_k = 1 + hF_y\,(x_k, \xi_k)$$

$$1 + r + r^2 + \cdots + r^m = \frac{r^{m+1}-1}{r-1}$$

$$y' = f\,(x,y) \quad \Rightarrow \quad y''' = f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2$$

END OF QUESTION PAPER