UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MODULE: PHY002Y

CAMPUS: APK

EXAM: NOVEMBER 2014

DATE: 20/11/2014	**SESSION 08:30 – 11:30**
ASSESSOR:	**DR E CARLESCHI**
MODERATOR:	**PROF S RAZZAQUE**
DURATION: 3 HOURS	**MARKS: 90**

NUMBER OF PAGES: 3 PAGES (excluding this information page)

INSTRUCTIONS:

1. Answer ALL the questions.
2. Programmable calculators are not permitted.
3. Remember that, in derivations, explanations carry marks as well as equations.

Question 1 [10 marks]

(1.1) Consider two equal but opposite charges $\pm q_1$, separated by a distance $2a_0$, that form a dipole that rotates with an angular frequency ω about its centre. A "receiver" single charge q_2 is located at a distance r_0 (with $a_0 \ll r_0$) from the centre of the dipole. Taking the retardation (or time lag) imposed by the theory of relativity into consideration, show that the electric field $E(t)$ experienced by q_2 at a time t is given by

$$E(t) = \frac{2q_1}{4\pi\epsilon_0 r_0^3}\, a\left(t - \frac{r_0}{c}\right),$$

where c is the speed of light. Important: Explain all steps and symbols used, and make use of a diagram if needed. *[7 marks]*

(1.2) Explain the difference between transverse and longitudinal wave motions, and give at least one example for each. *[3 marks]*

Question 2 [19 marks]

(2.1) Consider a thin, flexible string of mass per unit length M and subject to a tension W. By considering the net force acting on an element of the string, derive the wave equation governing its transverse motion

$$\frac{\partial^2 y}{\partial t^2} = \frac{W}{M}\frac{\partial^2 y}{\partial x^2},$$

where the wave displacement at time t and position x is given by $y(x,t)$. Important: Explain all steps and symbols used, and make use of a diagram. *[5 marks]*

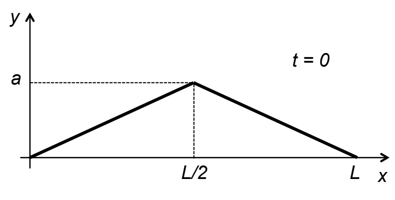
(2.2) Show that travelling waves of the form

$$y(x,t) = y(u),$$

where $u = x - vt$ and v is the speed of propagation of the wave, may be solutions of the wave equation in question (2.1), and find the two possible values of v that make them so. *[6 marks]*

(2.3) Write down the general solution comprising an arbitrary superposition of the two solutions found in question (2.2). *[1 mark]*

(2.4) The string of a harp is plucked by pulling it from its midpoint so that it has the profile shown below at time $t = 0$, when it is released from rest. Determine the motion of the harp string following its release from rest at $t = 0$. *[7 marks]*



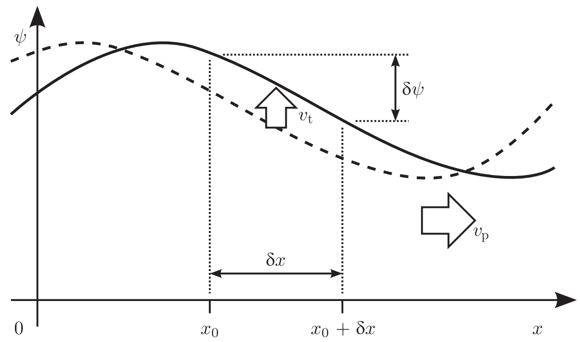
Question 3 [20 marks]

(3.1) **(a)** Why are sinusoidal waves so often considered? *[2 marks]*
(b) How are they related to complex exponential waves? *[2 marks]*

(3.2) A sinusoidal wave of frequency 40 Hz has a speed of propagation of 120 m/s.
(a) Calculate the wave period, wavelength and angular frequency. *[3 marks]*
(b) How far apart are two points whose displacements at any time differ in phase by 15°? *[2 marks]*
(c) At a given point, calculate the phase difference between two displacements occurring at times separated by 5 ms. *[2 marks]*

(3.3) The figure below shows a section of a guitar string undergoing a travelling wave motion $\psi(x,t) = \psi(u)$, where $u = x - v_p t$. Let us assume that the tension W is constant throughout the string (i.e. W is not affected by the presence of the wave), and that the wave amplitude is small in comparison with its wavelength. Show that the total energy $\mathcal{E}$ of a short section of the string of natural length δx, position x_0 and mass $\delta m = M\delta x$ (where M is the string's mass per unit length), is given by: *[9 marks]*

$$\mathcal{E} = W\left(\frac{\partial \psi}{\partial x}\right)^2 \delta x.$$



Question 4 [16 marks]

The wave equation governing the transverse motion for a skipping rope is the following:

$$M\frac{\partial^2 y}{\partial t^2} = W\frac{\partial^2 y}{\partial x^2} - \gamma\frac{\partial y}{\partial t},$$

where M is the linear mass density, W is the tension in the string, γ is a damping coefficient and $y(x,t)$ is the wave displacement at time t and position x.

(4.1) Show that the following complex exponential waveform

$$y(x,t) = a \exp i(kx - \omega t)$$

(where i is the imaginary unit and a is a constant) can be a solution of the wave equation above, provided that the following condition holds: $k^2 W = \omega^2 M + i\gamma\omega$. *[4 marks]*

(4.2) Assuming ω to be real and k to be complex, solve the equation $k^2 W = \omega^2 M + i\gamma\omega$ for k and calculate the expressions for the real and the imaginary part of k assuming that $\gamma \ll M\omega$. At the end of this derivation show that the complex exponential waveform in question (4.1) can be re-written as

$$y(x,t) = a \exp i(k_0 x - \omega t) \exp(-k_0 q x)$$

where $k_0 = \omega\sqrt{\dfrac{M}{W}}$ and $q = \dfrac{\gamma}{2M\omega}$. *[12 marks]*

Question 5 [10 marks]

Let's consider a particle that is trapped inside a one-dimensional box with infinitely tall walls (which is also called a one-dimensional rigid box, or infinite square well). The well's width goes from $x = 0$ to $x = a$. The particle of mass m moves inside the box at a non-relativistic speed, and there are no forces acting on the particle.

(5.1) Write down the general form for the particle's wave function $\Psi(x,t)$, and show how, by imposing the proper boundary conditions for the wave function, you can get to the quantisation of the wavelength λ of the particle inside the box. *[6 marks]*
(Please note: the wavelength is defined as $\lambda = 2\pi/k$, where k is the wavenumber.)

(5.2) Combine de Broglie relation $p = h/\lambda$ (where h is the Planck's constant) with the result for the wavelength in (5.1) in order to get to the quantisation of the energy for the particle in the box. *[4 marks]*

Question 6 [15 marks]

(6.1) The capillary-gravity wave equation for waves in shallow water is given by the following:

$$\frac{\partial^2 h}{\partial t^2} = gh_0 \frac{\partial^2 h}{\partial x^2} - \frac{h_0 \sigma}{\rho} \frac{\partial^4 h}{\partial x^4}$$

where g is the acceleration of gravity, h_0 is the water depth, σ is the surface tension, ρ is the density of water and $h(x,t)$ is the water displacement at time t and position x. Show that a sinusoidal waveform $h(x,t) = h_1 \sin(ku)$ (where $u = x - vt$) may be a solution of this wave equation, and find the expression for v as a function of k that makes it so. *[5 marks]*

(6.2) The Korteweg - de Vries wave equation:

$$\frac{\partial \psi}{\partial t} + 6\psi \frac{\partial \psi}{\partial x} + \frac{\partial^3 \psi}{\partial x^3} = 0$$

describes the motion of a soliton, i.e. a single wave pulse ψ that maintains its shape while it travels at constant speed. Show that the travelling wave pulse (with propagation speed c) given by the following:

$$\psi(x,t) = 2\alpha^2 \mathrm{sech}^2[\alpha(x - ct)]$$

may be a solution of the Kortewegde-Vries wave equation, and find the particular values of the parameter α that make it so. *[10 marks]*

HINT: To make the notation lighter, it is suggested that you use the variable $z = \alpha(x - ct)$ while calculating the time and space derivatives. Moreover, you might need the following information:

$\mathrm{sech}x = \dfrac{1}{\mathrm{cosh}x}$; $\dfrac{\mathrm{d}}{\mathrm{d}x}\mathrm{cosh}x = \mathrm{sinh}x$; $\dfrac{\mathrm{d}}{\mathrm{d}x}\mathrm{sinh}x = \mathrm{cosh}x$.

END of QUESTION PAPER