Question 1 [5]

Prove the Cauchy-Schwarz Inequality:
If $\overline{u}$ and $\overline{v}$ are vectors in an inner product space V, then

$$|\langle \overline{u}, \overline{v} \rangle| \leq \|\overline{u}\| \|\overline{v}\|.$$

Question 2 [2]

Prove that every orthogonally diagonalizable matrix is symmetric.

Question 3 [3]

Consider the theorem:
Every real n-dimensional vector space V is isomorphic to $\mathbb{R}^n$.

(a) Define the transformation $T : V \to \mathbb{R}^n$ used to prove this theorem. (1)

(b) Prove that T (defined in (a)) is linear. (2)

Question 4 [6]

Determine whether the following statements are TRUE or FALSE. Motivate the statement if TRUE; provide a counter-example if FALSE.

(a) If λ is an eigenvalue of A, then $-\lambda$ is an eigenvalue of $-A$. (2)

(b) If $\det(A) = \pm 1$, then A is an orthogonal matrix. (2)

(c) If $T : V \to V$ is a linear transformation and V is infinite-dimensional, then the rank of T is infinite. (2)

Question 5 [4]

Let $A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 4 & a \\ -3 & a & b \end{bmatrix}$, and suppose that A has two eigenvalues.

(a) Exactly one of the following vectors is not an eigenvector of A, the other two are eigenvectors of A. Determine the vector that is not an eigenvector of A. (2)

$$(1, 0, 1), \quad (1, 1, 0), \quad (-1, 0, 1).$$

(b) Hence, determine the eigenvalues of A, as well as the values of a and b. (2)

Question 6 [5]

Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\overline{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(a) Explain why $A\overline{x} = \overline{b}$ has a unique least squares solution. (1)

(b) Determine $\text{proj}_W \overline{b}$, where W is the column space of A viewed as a subspace of $\mathbb{R}^3$ with the following inner product: (4)

$$\langle (a_1, a_2, a_3), (b_1, b_2, b_3) \rangle = a_1 b_1 + 3a_2 b_2 + a_3 b_3.$$

Question 7 [2]

Consider M_{22} and let $A, B \in M_{22}$. Prove or disprove that the following defines an inner product on M_{22}

$$\langle A, B \rangle = \mathrm{tr}(AB).$$

Question 8 [3]

Give an example of the following (if such an example exists)

(a) A matrix consisting of real entries, but with no real eigenvalues. (1)

(b) An orthogonally diagonalizable matrix that is not an orthogonal matrix. (1)

(c) A matrix with orthonormal column vectors that is not an orthogonal matrix. (1)

Question 9 [9]

Suppose that $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and the characteristic polynomial for A is $(\lambda + 1)^2(\lambda - 4)$.

(a) Determine a basis for the eigenspace corresponding to $\lambda = -1$. (3)

(b) Given that $\{(2, 1, 0)\}$ is a basis for the eigenspace corresponding to $\lambda = 4$. Determine P and D that orthogonally diagonalize A. (3)

(c) Determine A^{10}. (3)

Question 10 [3]

Consider the quadratic form $x^2 + y^2 + 4xy = 1$.

(a) Determine whether the quadratic form is positive definite, negative definite, or indefinite. Show all calculations. (2)

(b) Hence, is this conic section a hyperbola, ellipse or neither? Explain. (1)

Question 11 [6]

Let $T : M_{22} \to \mathcal{P}_1$ be defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+d) + (b-c)x.$$

(a) Determine a basis for the kernel of T. (3)

(b) Hence, is T onto $\mathcal{P}_1$? Explain. (1)

(c) Let $S : \mathcal{P}_1 \to M_{22}$ be defined by $S(a+bx) = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$. Determine a formula for $S \circ T$ (if it exists). (2)

Question 12 [2]

Suppose that $T : V \to W$ has the following matrix transformation

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Is T onto W? Explain.