University of Johannesburg

Faculty of Science

**DEPARTMENT OF PURE AND APPLIED MATHEMATICS**

**MODULE** **MAT0AB2**
ENGINEERING LINEAR ALGEBRA B

**CAMPUS** **APK**

**EXAM** **NOVEMBER 2015**

Date 12/11/2015 Session 12:30 – 14:30

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Duration 2 Hours 60 Marks

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Surname and initials: ____________________

Student number: ____________________

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INSTRUCTIONS:

1. The paper consists of **12** printed pages, **excluding** the front page.
2. Read the questions carefully and answer all questions.
3. **Write out all calculations (steps) and motivate all answers.**
4. Questions are to be answered on the question paper in the space provided. Please indicate when the blank side of a page is used.
5. **No calculators are allowed.**

Question 1 [6]

Answer the following multiple choice questions in the **table given below**. Choose only **one** answer for each question.

(a) Which of the following maps $T$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a matrix transformation. (1)

A) $T(x_1, x_2) = (1 + x_1, x_2)$
B) $T(x_1, x_2) = (x_1 + x_2, x_2)$
C) $T(x_1, x_2) = (x_1^2, x_2)$
D) $T(x_1, x_2) = (\sin x_1, x_2)$
E) None of the above.

(b) Given two similar matrices $A$ and $B$. Which of the following statements is **FALSE**? (1)

A) If $A$ is invertible, then $A^{-1}$ is similar to $B$.
B) $A$ and $B$ have the same determinant.
C) $A$ and $B$ have the same eigenvalues.
D) $A$ and $B$ have the same trace.
E) None of the above.

(c) If $W$ is a subspace of an inner product space V. Which of the following statements is **FALSE**? (1)

A) $W^\perp$ is a subspace of $V$
B) $W \cap W^\perp = \{0\}$
C) $\mathsf{dim} W = \mathsf{dim} W^\perp$
D) $(W^\perp)^\perp = W$
E) None of the above.

(d) Select a matrix $A$ such that the orthogonal projection of a vector $\bar{b}$ on $W$, the column space of $A$, is given by (1)

$$\mathsf{proj}_W \bar{b} = A(A^T A)^{-1} A^T \bar{b}.$$

A) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.

B) $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$.

C) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$.

D) $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}$.

E) None of the above.

(e) Which of the following matrices is **not** orthogonally diagonalizable? (1)

A) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

B) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

C) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 3 \end{bmatrix}$

D) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

E) None of the above.

(f) Which of the following statements is **TRUE**? (1)

A) If $A$ is invertible, then $A$ is orthogonal.

B) If $A$ is orthogonal, then $A$ is invertible.

C) If $A$ is orthogonal, then $\mathsf{det}(A) = 1$.

D) If $A$ is orthogonal, then 0 is an eigenvalue of $A$.

E) None of the above.

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|---|---|---|---|---|---|
| Q1(a) | A | B | C | D | E |
| Q1(b) | A | B | C | D | E |
| Q1(c) | A | B | C | D | E |
| Q1(d) | A | B | C | D | E |
| Q1(e) | A | B | C | D | E |
| Q1(f) | A | B | C | D | E |

Question 2 [4]

Let $W$ be the plane in $\mathbb{R}^3$ (with the Euclidean inner product) with equation

$$2x - y - 4z = 0.$$

(a) Find a basis for $W$. (2)

(b) Find a basis for $W^{\perp}$. (2)

Question 3 [6]

Let $B$ be a $3 \times 3$ matrix with $-1$ as its only eigenvalue.

(a) $\mathsf{det}(-I - B)$=________________________________________ (1)

(b) Is $B$ invertible? **Motivate.** (1)

(c) What are the possible dimensions of the eigenspace corresponding to $\lambda = -1$? **Explain.** (2)

(d) What condition must be satisfied for $B$ to be diagonalizable? **Motivate.** (1)

(e) What are the eigenvalues of $B^4$. (1)

Question 4 [8]

Consider the bases $B = \{1 + 2x, 3 - x\}$ and $B' = \{1 - x, 1 + x\}$ for $P_1$, and let $\overline{p} = 5 - 4x$.

(a) Find the coordinate vector $[\overline{p}]_B$. (2)

(b) Find the transition matrix from $B$ to $B'$. (2)

(c) Find $[\overline{p}]_{B'}$ using the transition matrix found in $(b)$. (1)

Recall that $B = \{1 + 2x, 3 - x\}$.

(d) Let the vector space $P_1$ have the inner product $\langle \overline{p}, \overline{q} \rangle = 2a_0b_0 + 3a_1b_1$ where $\overline{p} = a_0 + a_1x$ and $\overline{q} = b_0 + b_1x$.
Show that $B$ is an orthogonal basis for $P_1$, but not orthonormal. Transform $B$ into an orthonormal basis for $P_1$. (3)

Question 5 [7]

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

(a) Verify that $A$ has a $QR$-decomposition. (1)

Recall that $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

(b) Find the $QR$-decomposition of the matrix $A$. (6)

Question 6 [5]

Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$ and $\overline{b} = \begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix}$.

(a) Find the least squares solution of the linear equation $A\overline{x} = \overline{b}$. (3)

(b) Find the least squares error vector resulting from the least squares solution $\overline{x}$. (2)

Question 7 [4]

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the matrix transformation defined by

$$T(x_1, x_2) = (x_1 + x_2, -x_1 + 2x_2).$$

(a) Show that $T$ is one-to-one. (1)

(b) Find the standard matrix $[T^{-1}]$ of the inverse of $T$. (2)

(c) Determine the formula $T^{-1}(\omega_1, \omega_2)$ for the inverse of $T$. (1)

Question 8 [2]

Verify that $\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$ is an orthogonal matrix and find its inverse.

Question 9 [9]

Consider the quadratic form $Q = 3x^2 - 4xy$, expressed in matrix notation as

$$\overline{x}^T A \overline{x} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

(a) Orthogonally diagonalize $A = \begin{bmatrix} 3 & -2 \\ -2 & 0 \end{bmatrix}$. (6)

(b) Classify the quadratic form $Q$ as positive definite, negative definite or indefinite. **Motivate**. (1)

Recall that $Q = 3x^2 - 4xy$ or in matrix form $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

(c) Find an orthogonal change of variable that eliminates the cross product terms in $Q$ and express $Q$ in terms of the new variables. (1)

(d) Identify the conic section represented by the equation $3x^2 - 4xy - 1 = 0$. (1)

Question 10 [4]

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix}$.

(a) Find an $LU$-decomposition of $A$. (3)

(b) Hence, find $A$'s unique $LDU$-decomposition. (1)

Question 11 [5]

Find a singular value decomposition of the matrix $A = \begin{bmatrix} 0 & 0 \\ 0 & 3 \\ -2 & 0 \end{bmatrix}$.