

QUESTION 1**[25]**

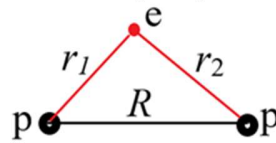
a) Determine an upper bound of the ground state energy for the delta function potential

$$V(x) = -\alpha\delta(x), \text{ using the normalised test function } \Psi = \left(\frac{2b}{\pi}\right)^{1/4} \exp(-bx^2)$$

$$\text{Hint: you are given, and do not need to prove, that } \langle \Psi | T | \Psi \rangle = \frac{b\hbar^2}{2m} \quad (8)$$

b) Prove that for the first excited state, the upper limit to its energy E_f is determined by $E_f \leq \langle \Psi | \hat{H} | \Psi \rangle$ if $\langle \Psi | \psi_g \rangle = 0$ (where ψ_g is the ground state wave function). (7)

c) The solitary electron in a singly ionised hydrogen molecule is illustrated below.



r_1 and r_2 are the distances of the electron to each of the two atomic nuclei.

i) Give and briefly explain the Hamiltonian of this system. (3)

ii) When applying the variational method with a test function $\Psi = A(\psi_0(r_1) + \psi_0(r_2))$, where ψ_0 is the standard wavefunction for the hydrogen atom ground state (with a corresponding energy E_0), show that the Schrödinger equation reduces to

$$\hat{H}\Psi = E_0\Psi - A \frac{e^2}{4\pi\epsilon_0} \left(\frac{\psi_0(r_2)}{r_1} + \frac{\psi_0(r_1)}{r_2} \right). \quad (7)$$

QUESTION 2**[25]**

a) In the so-called ‘classical region’, where a particle’s energy $E > V$ and $p(x)$ is real, the wave function ψ that is a solution to the Schrödinger equation can here be expressed as

$$\psi(x) = A(x)\exp(i\phi(x))$$

Confirm that this leads to the following relationships between A , ϕ and p :

$$\frac{d^2 A}{dx^2} = A \left[\left(\frac{d\phi}{dx} \right)^2 - \frac{p^2}{\hbar^2} \right] \text{ and } \frac{d(A^2(d\phi/dx))}{dx} = 0. \quad (10)$$

b) i) Draw the potential function for alpha particles in the vicinity of an atomic nucleus.

ii) How does quantum tunneling cause the alpha particle to escape the nucleus under some circumstances in what is referred to as alpha decay? (5)

c) Consider a particle of mass m in the following potential: $V(x) = \begin{cases} A \ln x + B & x > 0 \\ \infty & x < 0 \end{cases}$

Use the WKB approximation to determine the energy levels of this system. In particular, show that the difference between successive energy levels is independent of m and B .

$$[\text{Note: } \int_0^\infty \sqrt{x} e^{-x} dx = \frac{1}{2} \sqrt{\pi}] \quad (10)$$

QUESTION 3**[25]**

a) A two-level system can be described by the wave equation

$$\Psi(t) = c_a(t)\psi_a \exp\left(-i\frac{E_a}{\hbar}t\right) + c_b(t)\psi_b \exp\left(-i\frac{E_b}{\hbar}t\right).$$

What is the relationship between c_a and c_b , and why is that relationship justified. (2)

b) Consider a two-level system which is in the ground state at time $t = 0$, and where

$$H'_{ba}(t) = \exp(-(b + i\omega_0)t) ; H'_{ab}(t) = \exp((b + i\omega_0)t)$$

i) Calculate to first order the probability that the system will be in the excited state. (6)

ii) Describe, without doing actual calculations, how you would determine to second order the probability that the system will be in the ground state. (3)

c) In a two-level system, the equation relating the populations N of ground state “a” and excited state “b” is

$$\frac{dN_b}{dt} = -N_b A - N_b B_{ba} \rho(\omega_0) + N_a B_{ab} \rho(\omega_0)$$

where ρ is the radiation field density, and A , B_{ba} and B_{ab} are the Einstein coefficients.

Through comparison with the Planck and Boltzmann laws (both listed under the formulas), confirm that

$$B_{ab} = B_{ba} \quad \text{and} \quad A = \frac{\omega^3 \hbar}{\pi^2 c^3} B_{ba} \quad (9)$$

d) i) Describe the selection rules governing atomic transitions in terms of the quantum numbers l and m .

ii) What are the basic assumptions under which the selection rules $\Delta l = \pm 1$, $\Delta m = 0, \pm 1$ were determined. In other words, under what conditions are they valid. (6)

QUESTION 4**[25]**

a) Define the cross section σ , and hence determine its value for a series of small, hard spherical particles of radius r scattering with a larger hard sphere of radius R . (4)

b) An incident particle moving towards the z -direction and scattered in a direction defined by the angles θ and ϕ with a scattering amplitude $f(\theta, \phi) = \frac{1}{k} \sum_{l,m} (-i)^{l+1} C_{l,m} Y_l^m(\theta, \phi)$.

Hence prove that the cross section $\sigma = \frac{1}{k^2} \sum_{l,m} |C_{l,m}|^2$. (6)

c) Given that for $|\mathbf{r}| \gg |\mathbf{r}_0|$, the scattering amplitude may be approximated by

$$f(\theta, \phi) \cong -\frac{m}{2\pi\hbar^2} \int \exp(-i\mathbf{k} \cdot \mathbf{r}_0) V(\mathbf{r}_0) \psi(\mathbf{r}_0) d^3\mathbf{r}_0$$

i) Describe the Born approximation. (2)

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ii) Now invoke this approximation, and hence show that when the energy of the incoming and scattered particle are small, the scattering amplitude reduces to

$$f(\theta, \phi) \cong -\frac{m}{2\pi\hbar^2} \int V(\mathbf{r}) d^3\mathbf{r} . \quad (5)$$

iii) Hence confirm that the total cross section in the case of low-energy soft sphere scattering

$$V(r) = \begin{cases} V_0 & r < a \\ 0 & r > a \end{cases} \quad \text{equals} \quad \sigma \cong 4\pi \left(\frac{2mV_0a^3}{3\hbar^2} \right)^2 . \quad (8)$$

END