Supp Exam

(1) $(y^2\cos x - 3x^2y - 2x)dx + (2y\sin x - x^3 + \ln y)dy = 0$

$M(x,y) = y^2\cos x - 3x^2y - 2x$, $N(x,y) = 2y\sin x - x^3 + \ln y$

$\Rightarrow \dfrac{\partial M}{\partial y} = 2y\cos x - 3x^2$, $\dfrac{\partial N}{\partial x} = 2y\cos x - 3x^2$

$\therefore \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ and so the DE is exact. ✓

By this, $\exists$ a function $f = f(x,y)$ such that

$\dfrac{\partial f}{\partial x} = M = y^2\cos x - 3x^2y - 2x$ and ✓

$\dfrac{\partial f}{\partial y} = N = 2y\sin x - x^3 + \ln y$

Then * $f(x,y) = \int \dfrac{\partial f}{\partial x}dx = \int(y^2\cos x - 3x^2y - 2x)\,dx$

$= y^2\sin x - x^3y - x^2 + \varphi(y)$

$\Rightarrow \dfrac{\partial f}{\partial y} = 2y\sin x - x^3 + \varphi'(y) = 2y\sin x - x^3 + \ln y$ ✓

$\Rightarrow \varphi'(y) = \ln y$

$\Rightarrow \varphi(y) = y\ln y - y + k$ ✓

$\therefore$ ~~At~~ The general solution is

$y^2\sin x - x^3y - x^2 + y\ln y - y = c$ ✓

OR

** $f(x,y) = \int \dfrac{\partial f}{\partial y}dy = \int(2y\sin x - x^3 + \ln y)\,dy$

$= y^2\sin x - x^3y + y\ln y - y + \psi(x)$

$\dfrac{\partial f}{\partial x} = y^2\cos x - 3x^2y + \psi'(x) = y^2\cos x - 3x^2y - 2x$

$\Rightarrow \psi'(x) = -2x \Rightarrow \psi(x) = -x^2$

$\therefore$ The g.s. is

$y^2\sin x - x^3y + y\ln y - y - x^2 = c$

(2) $\left(x^3+3xy^2\right)\frac{dy}{dx}=y^3+3x^2y$; $y(2)=1$

$\Rightarrow \frac{dy}{dx}=\frac{y^3+3x^2y}{x^3+3xy^2}=F(y/x)$ (*)

$\therefore$ Homogeneous.

Let $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$ ✓

Then (*) becomes

$$v+x\frac{dv}{dx}=\frac{v^3x^3+3x^3v}{x^3+3x^3v^2}=\frac{v^3+3v}{1+3v^2}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v^3+3v}{1+3v^2}-v$$

$$=\frac{v^3+3v-v-3v^3}{1+3v^2}$$

$$=\frac{2v-2v^3}{1+3v^2} \checkmark$$

$$\Rightarrow \frac{1+3v^2}{v-v^3}\,dv=\frac{dx}{x}$$

$$\Rightarrow \left(\frac{1}{v}+\frac{2}{1-v}-\frac{2}{1+v}\right)dv=\frac{dx}{x}$$

$$\Rightarrow \ln|v|-2\ln|1-v|-2|1+v|=\ln|x|+C$$

$$\Rightarrow \ln\left|\frac{v}{(1-v^2)^2}\right|=\ln|x|+C$$

$$\Rightarrow \frac{v}{(1-v^2)^2}=Ax$$

$$\Rightarrow \frac{y/x}{(1-y^2/x^2)^2}=Ax \checkmark$$

$$y(2)=1 \Rightarrow \frac{1/2}{(1-1/4)^2}=2A \Rightarrow A=\frac{4}{9} \checkmark$$

$$\Rightarrow 9x^2y=4(x^2-y^2)^2 \checkmark$$

3) $x\frac{dy}{dx} + y = x^3y^2 \Rightarrow \frac{dy}{dx} + \frac{y}{x} = x^2y^2$ ✓

This is a Bernoulli equation with $n = 2$

Multiply by y^{-2}: $y^{-2}\frac{dy}{dx} + \frac{y^{-1}}{x} = x^2$ (#)

Let $v = y^{-1} \Rightarrow \frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$ ✓

Then (#) becomes

$$-\frac{dv}{dx} + \frac{v}{x} = x^2$$

$$\Rightarrow \frac{dv}{dx} - \frac{v}{x} = -x^2 \checkmark$$

This is a linear DE with integrating factor

$$f(x) = e^{\int(-1/x)dx} = e^{-\ln x} = 1/x \checkmark$$

$\therefore$ Solution is

$$\frac{1}{x}\cdot v = \int \frac{1}{x}(-x^2)\,dx$$

$$\Rightarrow \frac{v}{x} = \int (-x)\,dx = -\frac{x^2}{2} + C \checkmark$$

$$\Rightarrow \frac{y^{-1}}{x} = \frac{-x^2}{2} + C$$

$$y^{-1} = \frac{-x^3}{2} + Cx$$

$$\therefore y = \frac{2}{2Cx - x^3} \checkmark$$

4) (a) $\frac{dT}{dt} \propto (T - T_a) \Rightarrow \frac{dT}{dt} = k(T - T_a)$ ✓

T is the ~~temp~~ the object at time t, T_a is the ambient temperature and k is the proportionality constant

b) $T(0) = 24°C$, $T_a = -7°C$, $T(10) = -1°C$

Now, from (a) $\frac{dT}{T+7} = k\,dt \Rightarrow \ln|T+7| = kt + C$

$$\Rightarrow T = Ae^{kt} - 7 \checkmark$$

$T(0) = 24 \Rightarrow 24 = A - 7 \Rightarrow A = 31$ ✓

$$\Rightarrow T = 31e^{kt} - 7$$

$T(10) = -1 \Rightarrow -1 = 31e^{10k} - 7$ ✓

$\Rightarrow \quad k = \frac{1}{10}\ln\left(\frac{6}{31}\right) = -0.164$ ✓ (4)

$\therefore \quad T = 31e^{-0.164t} - 7$

15 minutes after the soft drink was put in the freezer, the temperature is $T(15) = 31e^{-0.164 \times 15} - 7 = -4.35°C$ ✓

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Question 5

a) $$f(t) = \begin{cases} -1 & , \quad 0 \leq t < 1 \\ 2t & \quad 1 \leq t < 3 \\ 4 & \quad t \geq 3 \end{cases}$$ ✓✓ ✓

b) $$f(t) = -1\left(u(t) - u(t-1)\right) + 2t\left(u(t-1) - u(t-3)\right) + 4\,u(t-3)$$ ✓
$$= -u(t) + (2t+1)\,u(t-1) + (4-2t)\,u(t-3)$$ ✓

c) $$\mathcal{L}\{f(t)\} = \mathcal{L}\{-u(t) + (2t+1)u(t-1) + (4-2t)u(t-3)\}$$

- $\mathcal{L}\{-u(t)\} = \dfrac{-1}{s}$

Recall:
$$\mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$$

- $\mathcal{L}\{(2t+1)\,u(t-1)\}$

Here, $a = 1$ and $f(t) = 2t+1$
Hence, $f(t+1) = 2t+3$ ✓

$$\therefore \mathcal{L}\{(2t+1)\,u(t-1)\} = e^{-s}\mathcal{L}\{2t+3\}$$
$$= e^{-s}\left(\frac{2}{s^2} + \frac{3}{s}\right)$$

- $\mathcal{L}\{(4-2t)\,u(t-3)\}$

Here, $a = 3$ and $f(t) = 4-2t$
Hence, $f(t+3) = -2-2t$ ✓

$$\therefore \mathcal{L}\{(4-2t)\,u(t-3)\} = e^{-3s}\mathcal{L}\{4-2t\}$$
$$= 2e^{-3s}\left(-\frac{1}{s} - \frac{1}{s^2}\right)$$

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finally,

$$\mathcal{L}\{f(t)\} = \frac{-1}{s} + e^{-s}\left(\frac{2}{s^2}+\frac{3}{s}\right) - 2e^{-3s}\left(\frac{1}{s}+\frac{1}{s^2}\right)$$

Question 6

Soln: $$\mathcal{L}^{-1}\left\{\frac{2s+4}{2s^2+2s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2}{s^2+s+\frac{1}{2}}\right\}$$

Complete square: $s^2+s+\frac{1}{2} = \left(s+\frac{1}{2}\right)^2+\frac{1}{4}$

$$\therefore \mathcal{L}^{-1}\{\quad\} = \mathcal{L}^{-1}\left\{\frac{s+2}{\left(s+\frac{1}{2}\right)^2+\frac{1}{4}}\right\}$$

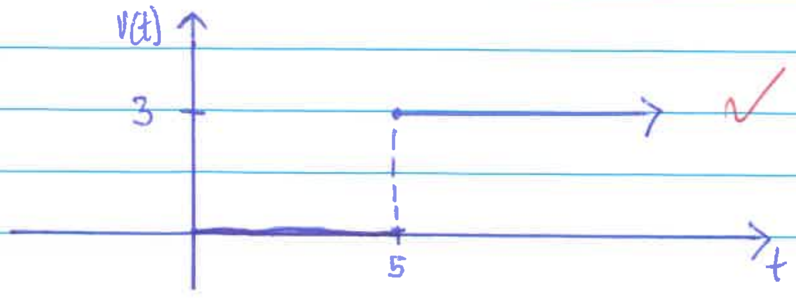
$$= \mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{1}{4}} + \frac{\frac{3}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{1}{4}}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{1}{4}} + 3 \cdot \frac{\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{1}{4}}\right\}$$

$$= e^{-\frac{1}{2}t}\left(\cos\frac{1}{2}t + 3\sin\frac{1}{2}t\right)$$

Question 7

Soln: $v(t) = 3\,u(t-5)$



Question 8

a) $y'' + 2y' - 3y = e^{-3(t-2)}u(t-2)$, $y(0) = y'(0) = 1$

Soln:

$$\mathcal{L}\{y'' + 2y' - 3y\} = \mathcal{L}\{e^{-3(t-2)}u(t-2)\}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 2(s\mathcal{L}\{y\} - y(0)) - 3\mathcal{L}\{y\} = \frac{e^{-2s}}{s+3}$$ ✓

$$(s^2 + 2s - 3)\mathcal{L}\{y\} = \frac{e^{-2s}}{s+3} + s + 3$$ ✓

$$\therefore \mathcal{L}\{y\} = \frac{e^{-2s}}{(s+3)(s+3)(s-1)} + \frac{s+3}{(s+3)(s-1)}$$

$$\therefore y = \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{(s+3)^2(s-1)} + \frac{1}{(s-1)}\right\}$$ ✓ 1/2

i.e. $\dfrac{1}{(s+3)^2(s-1)} = \dfrac{A}{s+3} + \dfrac{B}{(s+3)^2} + \dfrac{C}{s-1}$

$A = -1/16$ ✓ 1/2 , $B = -1/4$ ✓ 1/2 , $C = 1/16$ ✓ 1/2

$$\therefore y = \mathcal{L}^{-1}\left\{e^{-2s}\left(\frac{-1/16}{s+3} - \frac{\frac{1}{4}}{(s+3)^2} + \frac{\frac{1}{16}}{s-1}\right) + \frac{1}{s-1}\right\}$$

$$= \left[\frac{-1}{16}e^{-3(t-2)} - \frac{1}{4}(t-2)e^{-3(t-2)} + \frac{1}{16}e^{t-2}\right]u(t-2) + e^{t}$$ ✓ ✓ ✓ ✓

b) Stead-state ✓

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Question 9

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[12]

a $f(x) = \begin{cases} 2, & -2 \le x \le 0 \\ x & 0 \le x \le 2 \end{cases}$ $f(x) = f(x+4)$

(a)

$$a_0 = \frac{2}{4}\left[\int_{-2}^{0} 2\,dx + \int_{0}^{2} x\,dx\right]$$

$$= \frac{1}{2}\left[2x\Big|_{-2}^{0} + \frac{x^2}{2}\Big|_{0}^{2}\right] ✓$$

$$= \frac{1}{2}\left[4 + 2\right] = 3 ✓$$

$$a_n = \frac{1}{2}\left[\int_{-2}^{0} 2\cos\frac{2n\pi x}{4}\,dx + \int_{0}^{2} x\cos\frac{2n\pi x}{4}\,dx\right] ✓$$

$$= \frac{1}{2}\left[\frac{4}{n\pi}\sin\frac{n\pi x}{2}\Big|_{-2}^{0} + \frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\Big|_{0}^{2}\right] ✓$$

$$= \frac{1}{2}\left[\frac{4}{n\pi}\cancel{\sin 0}^{0} + \frac{4}{n\pi}\cancel{\sin n\pi}^{0} + \frac{4}{n\pi}\cancel{\sin n\pi}^{0} + \frac{4}{n^2\pi^2}\cos n\pi - \frac{4}{n^2\pi^2}\cos 0\right] ✓$$

$$= \frac{1}{2}\left[\frac{4}{n^2\pi^2}(\cos n\pi - 1)\right]$$

$$= \begin{cases} \dfrac{-4}{n^2\pi^2}, & n = \text{odd} ✓ \\ 0, & n = \text{even} ✓ \end{cases}$$

$$b_n = \frac{1}{2}\left[\int_{-2}^{0} 2\sin\frac{n\pi x}{2} + \int_0^2 x\sin\frac{n\pi x}{2}\,dx\right] ✓$$

$$= \frac{1}{2}\left[-\frac{4}{n\pi}\cos\frac{n\pi x}{2}\Big|_{-2}^{0} - \frac{2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\Big|_0^2\right] ✓$$

$$= \frac{1}{2}\left[\frac{-4}{n\pi} + \cancel{\frac{4}{n\pi}\cos n\pi} - \cancel{\frac{4}{n\pi}\cos n\pi} + \frac{4}{n^2\pi^2}\overset{0}{\cancel{\sin n\pi}}\right] ✓$$

$$= \frac{-2}{n\pi} ✓$$

$$f(x) = \frac{3}{2} + \frac{4}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{9}\cos\frac{3\pi x}{2} + \dots\right)$$

$$- \frac{2}{\pi}\left(\sin\frac{\pi x}{2} + \frac{1}{2}\sin\pi x + \frac{1}{3}\sin\frac{3\pi x}{2} + \frac{1}{4}\sin 2\pi x + \dots\right) ✓$$

(b) $c_n = \frac{1}{4}\int f(x)\, e^{-i\cdot\frac{2n\pi x}{P}}\,dx$ [6]

$$= \frac{1}{4}\left[\int_{-2}^{0} 2\cdot e^{-\frac{in\pi x}{2}}\,dx + \int_0^2 x e^{-\frac{in\pi x}{2}}\,dx\right] ✓$$

let $u = x$ $\quad dv = e^{-\frac{in\pi x}{2}}\,dx$

$du = dx$ $\quad v = \frac{2}{-in\pi} e^{-\frac{in\pi x}{2}}$

$$\therefore \frac{2x}{-in\pi} e^{-\frac{in\pi x}{2}} - \int \frac{2}{-in\pi} e^{-\frac{in\pi x}{2}}\,dx$$

$$\frac{2x}{-in\pi} e^{-\frac{in\pi x}{2}} + \frac{4}{n^2\pi^2} e^{-\frac{in\pi x}{2}}$$

$$\therefore c_n = \frac{1}{4}\left[\frac{4}{-in\pi} e^{-\frac{in\pi x}{2}}\Big|_{-2}^{0} + \frac{2ix}{n\pi} e^{-\frac{in\pi x}{2}} + \frac{4}{n^2\pi^2} e^{-\frac{in\pi x}{2}}\Big|_0^2\right] ✓$$

$$= \frac{1}{4}\left[\frac{4i}{n\pi}\left(1 - e^{in\pi}\right) + \left(\frac{4i}{n\pi} e^{-in\pi} + \frac{4}{n^2\pi^2} e^{-in\pi}\right) - \frac{4}{n^2\pi^2}\right] \checkmark$$

$$= \frac{1}{4}\left[\frac{4i}{n\pi} - \frac{4i}{n\pi} e^{in\pi} + \frac{4i}{n\pi} e^{-in\pi} + \frac{4}{n^2\pi^2} e^{-in\pi} - \frac{4}{n^2\pi^2}\right]$$

$$= \frac{i}{n\pi} - \frac{i}{n\pi}\left(e^{in\pi} - e^{-in\pi}\right) + \frac{1}{n^2\pi^2}\left(e^{-in\pi} - 1\right)$$

$$= \frac{i}{n\pi} - \frac{2i}{n\pi} \xcancel{\sin n\pi}^{\,0} + \frac{1}{n^2\pi^2}\left(e^{-in\pi} - 1\right) \checkmark$$

$$= \frac{i}{n\pi} + \frac{1}{n^2\pi^2}\left(e^{-in\pi} - 1\right) \checkmark$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty}\left(\frac{i}{n\pi} + \frac{1}{n^2\pi^2}\left(e^{-in\pi} - 1\right)\right) e^{\frac{in\pi x}{2}} \checkmark$$

Question 10

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx \qquad [4]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{2ix} \cdot e^{i\alpha x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{(2+\alpha)ix}\, dx \checkmark$$

$$= \frac{1}{\sqrt{2\pi}\,(2+\alpha)i}\, e^{(2+\alpha)ix}\Big|_{-1}^{1} \checkmark$$

$$= \frac{1}{\sqrt{2\pi}\,(2+\alpha)i}\left(e^{(2+\alpha)i} - e^{-(2+\alpha)i}\right) \checkmark$$

$$= \frac{2\sin(2+\alpha)}{\sqrt{2\pi}\,(2+\alpha)i} \checkmark$$