# Mathematics and Applied Mathematics

## Calculus of One Variable Functions
## Exam Solutions

### MAT2EB1\MAT1A2E: 04/11/2021

**Time:** 12H30—15H30 **Marks:** 35
**Assessors:** Mr. Chikore, Mr. Matsebula and Mrs. Sebogodi

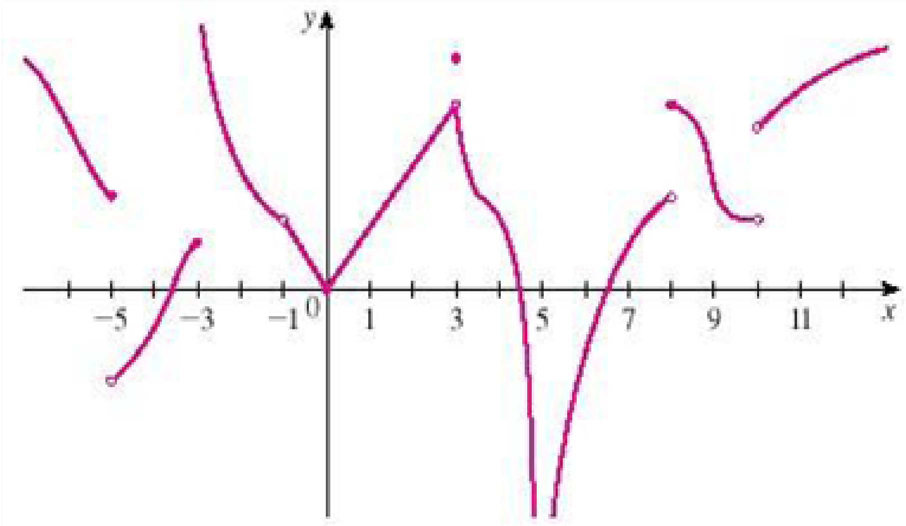


**Figure 1:** The graph of $f(x)$.

**Question 1 [10 mark(s)]**

Use Figure 1 to answer the following questions. Do not justify.

(a) Does the limit from the left of $f$ at $-5$ exist? (2)

(b) Does the limit from the right of $f$ at $-5$ exist? (2)

(c) Does the limit of $f$ at $-5$ exist? (2)

(d) Is $f$ continuous from the left at $-5$? (2)

(e) Is $f$ continuous from the right at $-5$? (2)

Solution

(a) Yes✓✓

(b) Yes✓✓

(c) No✓✓

(d) Yes✓✓

(e) No✓✓

**Question 2 [4 mark(s)]**

Find the derivative of the function

$y = \dfrac{2x}{4-x}$

**Solution**

$$\begin{aligned} y &= \frac{2x}{4-x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(4-x)\frac{\mathrm{d}}{\mathrm{d}x}(2x) - 2x\frac{\mathrm{d}}{\mathrm{d}x}(4-x)}{(4-x)^2} \checkmark\checkmark \\ &= \frac{2(4-x)+2x}{(4-x)^2} \checkmark \\ &= \frac{8}{(4-x)^2} \checkmark \end{aligned}$$

**Question 3 [17 mark(s)]**

Evaluate the following limits.

(a) (4)

$$\lim_{x\to\infty} \frac{4x^2+2}{x\sqrt{5x^2+1}} = \lim_{x\to\infty} \frac{4x^2+2}{x\underbrace{|x|}_{\checkmark}\sqrt{5+\underbrace{x^{-2}}_{\checkmark}}} = \lim_{x\to\infty} \frac{\overbrace{4+2x^{-2}}^{\checkmark}}{\sqrt{5+x^{-2}}} = \frac{4}{\sqrt{2}} \checkmark$$

(b) (4)

$$\lim_{x\to\infty} \frac{\sinh(3x)}{7e^{3x}} = \lim_{x\to\infty} \frac{\overbrace{e^{3x}-e^{-3x}}^{\checkmark}}{14e^{3x}} = \lim_{x\to\infty} \frac{\overbrace{1-e^{-6x}}^{\checkmark\checkmark}}{14} = \frac{1}{14} \checkmark$$

(c) (4)

$$\lim_{x\to 0} \frac{3^x-4^x}{x} = \lim_{x\to 0} \frac{\overbrace{3^x\ln 3}^{\checkmark} - \overbrace{4^x\ln 4}^{\checkmark}}{1} = \overbrace{\ln 3}^{\checkmark} - \overbrace{\ln 4}^{\checkmark}$$

(d) For the following question, use $f(x) = |2x-8|$ to evaluate the following limits.

$(I)$ (2)

$$\begin{aligned} \lim_{x\to 4^+} \frac{f(x)-f(4)}{x-4} &= \lim_{x\to 4^+} \frac{|2x-8|}{x-4} \\ &= \lim_{x\to 4^+} \frac{2|x-4|}{x-4} \\ &= \lim_{x\to 4^+} \underbrace{\frac{2(x-4)}{x-4}}_{\checkmark} = 2\checkmark \end{aligned}$$

$(II)$ (2)

$$\begin{aligned} \lim_{x\to 4^-} \frac{f(x)-f(4)}{x-4} &= \lim_{x\to 4^-} \frac{|2x-8|}{x-4} \\ &= \lim_{x\to 4^-} \frac{2|x-4|}{x-4} \\ &= \lim_{x\to 4^-} \underbrace{\frac{-2(x-4)}{x-4}}_{\checkmark} = -2\checkmark \end{aligned}$$

$(III)$ Does the value $f'(4)$ exist? Justify your answer. (1)

No, since $\lim\limits_{x\to 4^-} f \neq \lim\limits_{x\to 4^+} f$✓

**Question 4 [4 mark(s)]**

Use mathematical induction to prove the following proposition.

$$\sum_{i=0}^{n-3} 4^{i+3} = \frac{4}{3}(4^n - 16), \quad n \geq 3$$

**Solution**

Let $n = 3$

$$\text{LHS} = \sum_{i=0}^{0} 4^{i+3} = 4^3, \quad \text{RHS} = \frac{4}{3}(4^3-16) = 4^3 \therefore \text{LHS} \overset{\checkmark}{=} \text{RHS}.$$

We want to show that

$$\sum_{i=0}^{k-3} 4^{i+3} = \frac{4(4^k-16)}{3} \implies \sum_{i=0}^{k-2} 4^{i+3} = \frac{4(4^{k+1}-16)}{3}.$$

$$\begin{aligned}
\text{LHS} &= \sum_{i=0}^{k-2} 4^{i+3} \\
&= \sum_{i=0}^{k-3} 4^{i+3} + 4^{k+1} && \text{(Remove the last term from the sum)} \\
&= \frac{4(4^k-16)}{3} + 4^{k+1} && \text{(Hypothesis)} \\
&= \frac{4^{k+1} - 4(16) + 3(4^{k+1})}{3} && \text{(Simplify)} \\
&= \frac{4}{3}(4^{k+1}-16) = \text{RHS}
\end{aligned}$$

✓: half a mark.
✓: one mark.