JUNE EXAM MEMO VERSION 1 APK

Q 1

1.1 D

1.2 C

1.3 A

1.4 A

1.5 D

1.6 B

1.7 D

1.8 C

1.9 A

1.10 B

Q(2a)

$$y = \ln(1-x^2) \Rightarrow \frac{dy}{dx} = -\frac{2x}{1-x^2}$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = 1+\left(\frac{-2x}{1-x^2}\right)^2 = \left(\frac{1+x^2}{1-x^2}\right)^2$$

$$\Rightarrow L = \int_0^{1/4} \sqrt{\left(\frac{1+x^2}{1-x^2}\right)^2}\,dx$$

$$\therefore \boxed{L = \int_0^{1/4} \left(\frac{1+x^2}{1-x^2}\right) dx}$$

Q(2b)

$$y = \tfrac{1}{4}(x^2 - 2\ln x) \Rightarrow \frac{dy}{dx} = \tfrac{1}{4}\left(2x - \frac{2}{x}\right) = \frac{x}{2} - \frac{1}{2x}$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = 1+\left(\frac{x}{2}-\frac{1}{2x}\right)^2 = \left(\frac{x}{2}+\frac{1}{2x}\right)^2$$

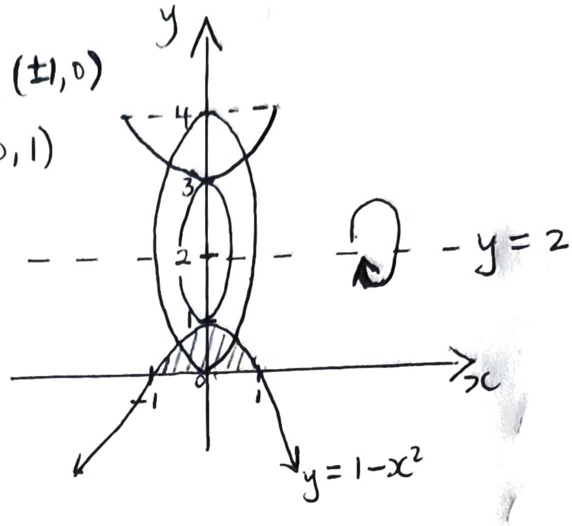
$$\Rightarrow S = 2\pi \int_1^4 x\sqrt{\left(\frac{x}{2}+\frac{1}{2x}\right)^2}\,dx = 2\pi\int_1^4 \left(\frac{x^2}{2}+\frac{1}{2}\right)dx$$

$$\Rightarrow S = \pi\int_1^4 (x^2+1)\,dx = \pi\left[\frac{x^3}{3}+x\right]_1^4 = \pi\left[\frac{76}{3}-\frac{4}{3}\right] = \boxed{24\pi}$$

Q3 $y = 1 - x^2$

x-intercept : $y = 0 \Rightarrow x = \pm 1 \therefore (\pm 1, 0)$

y-intercept : $x = 0 \Rightarrow y = 1 \therefore (0, 1)$



$r_{out} = y_T - y_b = 2 - 0 = 2$

$r_{in} = y_T - y_b = 2 - (1 - x^2) = 1 + x^2$

$$V = \pi \int_{-1}^{1} \left[2^2 - (1+x^2)^2 \right] dx$$

$$V = \pi \int_{-1}^{1} \left[4 - (1 + 2x^2 + x^4) \right] dx$$

$$\Rightarrow V = \pi \int_{-1}^{1} (3 - 2x^2 - x^4) dx$$

$$\Rightarrow V = \pi \left[3x - 2\frac{x^3}{3} - \frac{x^5}{5} \right]_{-1}^{1}$$

$$\Rightarrow V = \pi \left[\left(3 - \frac{2}{3} - \frac{1}{5}\right) - \left(-3 + \frac{2}{3} + \frac{1}{5}\right) \right]$$

$$\therefore \boxed{V = \frac{64\pi}{15}}$$

Q(4a) $\displaystyle\int_1^e \frac{\ln x}{x^2} dx$

$u = \ln x \qquad v' = \frac{1}{x^2}$

$u' = \frac{1}{x} \qquad v = -\frac{1}{x}$

$$\Rightarrow \int_1^e \frac{\ln x}{x^2} dx = \left[-\frac{\ln x}{x} \right]_1^e + \int_1^e \frac{1}{x^2} dx$$

$$\Rightarrow \int_1^e \frac{\ln x}{x^2} dx = -\frac{1}{e} + \left[-\frac{1}{x} \right]_1^e$$

$$\therefore \boxed{\int_1^e \frac{\ln x}{x^2} dx = 1 - \frac{2}{e}}$$

Q(4b)

$$\int_3^2 \frac{dx}{\sqrt{3-x}} = -\int_2^3 \frac{dx}{\sqrt{3-x}}$$

$u = 3 - x$
$du = -dx$

$x = 2 \to u = 1$
$x = t \to u = 3 - t$

$$-\int_2^3 \frac{dx}{\sqrt{3-x}} = \lim_{t \to 3^-} \int_2^t \frac{-dx}{\sqrt{3-x}}$$

$$= \lim_{t \to 3^-} \int_1^{3-t} u^{-\frac{1}{2}} du$$

$$= \lim_{t \to 3^-} \left[2\sqrt{u} \right]_1^{3-t}$$

$$= \lim_{t \to 3^-} \left[2\sqrt{3-t} - 2 \right]$$

$$\therefore \boxed{\int_3^2 \frac{dx}{\sqrt{3-x}} = -2}$$

Q(5a) $\frac{dy}{dx} = \frac{1+y^2}{(1+x^2)xy}$

$\Rightarrow \int \frac{y\,dy}{1+y^2} = \int \frac{dx}{x(1+x^2)}$

$\downarrow$

$u = 1+y^2$
$du = 2y\,dy$
$\frac{1}{2}du = y\,dy$

$\Rightarrow \frac{1}{2}\int \frac{du}{u} = \int \frac{1}{x} - \frac{x}{1+x^2}\,dx$

$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}$

$1 = A(1+x^2) + (Bx+C)(x)$

$1 = A(1+x^2) + Bx^2 + Cx$

$A+B = 0$ - - - ①
$C = 0$ - - - ②
$A = 1$ - - - ③
$\Rightarrow B = -1$

$\therefore A = 1,\ B = -1,\ C = 0$

$\Rightarrow \frac{1}{2}\ln|u| = \ln|x| - \frac{1}{2}\ln(1+x^2) + C$

$\Rightarrow \frac{1}{2}\ln|1+y^2| = \ln|x| - \frac{1}{2}\ln(1+x^2) + C$

$\Rightarrow \ln(1+y^2) = 2\ln x - \ln(1+x^2) + 2C,\ x > 0$

$\Rightarrow \ln(1+y^2) = \ln\left(\frac{x^2}{1+x^2}\right) + 2C$

$\Rightarrow 1+y^2 = e^{2C}\left(\frac{x^2}{1+x^2}\right)$

$\Rightarrow \boxed{y = \pm\sqrt{A\left(\frac{x^2}{1+x^2}\right) - 1}}$ where $A = e^{2C}$

Q(5b) $\frac{dy}{dx} + (\sec^2 x)y = \sec^2 x$

$\Rightarrow \frac{dy}{dx} = \sec^2 x\,(1-y)$

$\Rightarrow \int \frac{dy}{1-y} = \int \sec^2 x\,dx$

$\Rightarrow \ln|1-y| = \tan x + C$

$y(\frac{\pi}{4}) = 0 \Rightarrow C = -1$

$\Rightarrow \ln|1-y| = \tan x - 1$

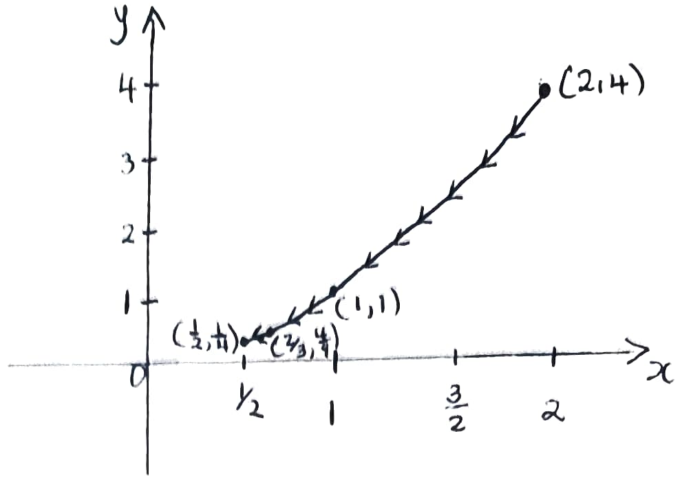
$\Rightarrow |1-y| = e^{-1}e^{\tan x}$

$\Rightarrow 1-y = \pm e^{-1}e^{\tan x}$

$\Rightarrow \boxed{y = 1 - Ae^{\tan x}}$ where $A = \pm\frac{1}{e}$

Q(6a)

t	x	y
1	2	4
2	1	1
3	$\frac{2}{3}$	$\frac{4}{9}$
4	$\frac{1}{2}$	$\frac{1}{4}$



Q(6b) $x'(t) = -\frac{2}{t^2}$, $y'(t) = -\frac{8}{t^3}$

$$L = \int_1^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\Rightarrow L = \int_1^4 \sqrt{\frac{4}{t^4} + \frac{64}{t^6}}\, dt$$

$$\therefore \boxed{L = \int_1^4 \frac{2}{t^3}\sqrt{t^2+16}\, dt}$$

Q(6c) $\frac{dx}{dt} = 6t$, $\frac{dy}{dt} = 6t^2$

$$S = \int_0^5 2\pi(3t^2)\sqrt{(6t)^2 + (6t^2)^2}\, dt$$

$$\Rightarrow S = \int_0^5 2\pi(3t^2)\sqrt{36t^2 + 36t^4}\, dt.$$

$$\therefore \boxed{S = 36\pi\int_0^5 t^3\sqrt{1+t^2}\, dt}.$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Q7 $$\left(\sqrt{y} - \frac{x}{3}\right)^4 = \sum_{k=0}^{4}\binom{4}{k}(\sqrt{y})^{4-k}\left(-\frac{x}{3}\right)^k$$

$$= \binom{4}{0}(\sqrt{y})^4\left(-\frac{x}{3}\right)^0 + \binom{4}{1}(\sqrt{y})^3\left(-\frac{x}{3}\right) + \binom{4}{2}(\sqrt{y})^2\left(-\frac{x}{3}\right)^2 + \binom{4}{3}(\sqrt{y})\left(-\frac{x}{3}\right)^3 + \binom{4}{4}\left(-\frac{x}{3}\right.$$

$$= (1)y^2 + 4y^{3/2}\left(-\frac{x}{3}\right) + 6(y)\left(\frac{x^2}{9}\right) + 4(y^{1/2})\left(-\frac{x^3}{27}\right) + (1)\left(\frac{x^4}{81}\right)$$

$$\therefore \boxed{\left(\sqrt{y} - \frac{x}{3}\right)^4 = y^2 - \frac{4}{3}xy^{3/2} + \frac{2}{3}x^2y - \frac{4}{27}x^3y^{1/2} + \frac{x^4}{81}}$$

Q8(a) $\begin{bmatrix} 3 & 0 & 0 & 5 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & -1 & 7 \end{bmatrix} \longrightarrow \begin{matrix} 3x = 5 \\ y + 4z = 0 \\ -2y - z = 7 \end{matrix}$

Q(8b) $\begin{matrix} x_1 - x_5 = 7 \\ x_2 = -1 \end{matrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 7 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$

Q(8c) $\begin{matrix} x + z = 6 \\ -3y + z = 7 \\ 2x + y + 3z = 15 \end{matrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 6 \\ 0 & -3 & 1 & 7 \\ 2 & 1 & 3 & 15 \end{bmatrix}$

$\downarrow R_3 = R_3 - 2R_1$

$\begin{bmatrix} 1 & 0 & 1 & 6 \\ 0 & 1 & 5 & 19 \\ 0 & 1 & 1 & 3 \end{bmatrix} \xleftarrow{R_2 = R_2 + 4R_3} \begin{bmatrix} 1 & 0 & 1 & 6 \\ 0 & -3 & 1 & 7 \\ 0 & 1 & 1 & 3 \end{bmatrix}$

$\downarrow R_3 = R_3 - R_2$

$\begin{bmatrix} 1 & 0 & 1 & 6 \\ 0 & 1 & 5 & 19 \\ 0 & 0 & -4 & -16 \end{bmatrix} \xrightarrow{R_3 = -\frac{1}{4}R_3} \begin{bmatrix} 1 & 0 & 1 & 6 \\ 0 & 1 & 5 & 19 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow[R_1 = R_1 - R_3]{R_2 = R_2 - 5R_3} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

$\therefore \boxed{(x, y, z) = (2, -1, 4)}$

Q9 $|B| = 4(2) - (-1)(0) = 8$

$\text{adj}(B) = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$

$B^{-1} = \frac{1}{8}\begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/8 \\ 0 & 1/2 \end{bmatrix}$

$B^{-1}C = \begin{bmatrix} 1/4 & 1/8 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 5/8 & 9/8 & 9/8 \\ 3/2 & 1/2 & 5/2 \end{bmatrix}$

$B^{-1}C = \frac{1}{K}\begin{bmatrix} 10 & 18 & 18 \\ 24 & 8 & 40 \end{bmatrix} \Rightarrow \frac{5}{8} = \frac{10}{K} \quad \therefore \boxed{K = 16}$