PHY8X09 – 2021 – Exam – answers and marking (NOT a rote learning solution set)

QUESTION 1 [30]

a) A particular star with a mass 0.140 times the mass of the Sun and known to be 1.75 pc away is seen to oscillate about its mean position with an amplitude of 0.025 arcseconds with a period of 26 years. The oscillation is believed to be caused by a planet in a near-circular orbit around this star. Determine the distance from the planet to the star and the planet's mass. [Hint: use as an approximation the fact that the mass of the star is much larger than the mass of the planet] [assignment] (8)

By Kepler's third law, … (after some calculations) … $\Rightarrow m_p = 2.70 \times 10^{27}$ kg

b) Show that for small wavelengths Planck's Law simplifies to Wien's Law

$$B_\lambda \simeq \frac{2hc^2}{\lambda^5} \exp\left(-\frac{hc}{\lambda kT}\right)$$

Furthermore confirm that this means that $B_\lambda \to 0$ as $\lambda \to 0$. [derivation, unseen] (4)

Proving this expression, and explaining why it tends to zero

c) Define the specific energy density, and confirm that this is equal to $u_\lambda = \frac{4\pi}{c}\langle I_\lambda \rangle$.

[conceptual, derivation] (7)

Full marks not only for the correct definition, but also for proving this identity

d) A photon is moving from its origin in a uniform medium, scattering at random in a random direction. If in time t it moves an average distance d from the origin, how long will it take, on average, to reach a distance $2d$ from the origin? [conceptual] (4)

$4t$. (with a full explanation)

e) Given the radiative transfer equation form $\cos\theta \frac{dI_\lambda}{d\tau_{\lambda,v}} = I_\lambda - S_\lambda$, show that, for a grey atmosphere, $\cos\theta \frac{dI}{d\tau_v} = I - S$, and hence that $\frac{dF_{rad}}{d\tau_v} = 4\pi\left(\langle I \rangle - S\right)$.

[I and S are defined by $I = \int_0^\infty I_\lambda d\lambda$ and $S = \int_0^\infty S_\lambda d\lambda$] . [derivation] (7)

If we integrate $\cos\theta \frac{dI_\lambda}{d\tau_{\lambda,v}} = I_\lambda - S_\lambda$ over all wavelengths,
… (after some calculations) … $\Rightarrow \frac{dF_{rad}}{d\tau_v} = 4\pi\left(\langle I \rangle - S\right)$

QUESTION 2 [20]

a) A star has a density whose dependence on the distance r to the centre of the star is given by

$$\rho(r) = \rho_0\left(1 - \frac{r^2}{R^2}\right)$$

where ρ_0 is the density at the star's centre, and R is the outer stellar radius.

Determine an expression for the mass inside a distance r from the star's centre in terms of ρ_0 and R. Hence obtain the expression for the total mass M of the star. [assignment] (7)

$dm(r) = \rho(r)dV = ...$, … (after some calculations) … $M = ... = \frac{8}{15}\pi\rho_0 R^3$

b) You wish to test a (wrong) theory that the energy of stars solely comes from the gravitational energy released while the interstellar cloud condenses.
Show that the total gravitational energy released due to this process for a star of mass M, radius R and uniform density would be given by

$$E \sim -\frac{3}{10}\frac{GM^2}{R}.$$ [derivation] (9)

Explaining properly that $dU = -G\frac{M_r dm}{r}$, (after some calculations) … $E = \frac{3}{10}G\frac{M^2}{R}$

c) Show that the criterion for convection $\left|\frac{dT}{dr}\right|_{actual} > \left|\frac{dT}{dr}\right|_{adiabatic} \left[= \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} \right]$

may equally be expressed as $\frac{d\ln P}{d\ln T} < \frac{\gamma}{\gamma - 1}$ [derivation] (4)

$\frac{d\ln P}{d\ln T} = ...$; … (after some calculations) … $\frac{\gamma}{\gamma-1} > \frac{d\ln P}{d\ln T}$

QUESTION 3 [25]

a) Comparing two stars, one with $B–V > 0$ and one with $B–V < 0$, which is the redder of the two, and why? [class example] (3)

The former. Explanation requires student to demonstrate understanding

b) We know that the radial speed of particles at the outer edge of a gravitationally collapsing interstellar cloud of initial radius r_0 and density ρ_0 is given by the expression

$$\frac{dr}{dt} = -\left[\frac{8\pi}{3}G\rho_0 r_0^2\left(\frac{r_0}{r} - 1\right)\right]^{1/2}$$

Describe the procedure and assumptions you would use to determine the time required to collapse the cloud to a star. NB: do not do any calculations. [derivation] (5)

Any correct description and interpretation of this equation …

c) What is the Hertzsprung-Russel (HR) diagram? Hence briefly discuss the evolution of stars with reference to this diagram. [conceptual] (10)

(No unique answer here. 2 marks for identifying the HR diagram as a plot of luminosity vs. temperature/colour. The remaining marks for relevant facts, ….)

d) Given that, assuming near-constant density, the pressure as a function of distance from a star's centre is given by $P(r) = \frac{2}{3}\pi G\rho^2\left(R^2 - r^2\right)$, and that the speed of sound in a gas with adiabatic ratio γ is $v_s = \sqrt{\frac{\gamma P}{\rho}}$, confirm that the period of oscillation for a pulsating star is approximately $\Pi \approx \sqrt{\frac{3\pi}{2\gamma G\rho}}$. [derivation] (7)

If a particle oscillates with a period Π, then … (after some calculations) $\therefore \Pi = \sqrt{\frac{3\pi}{2\gamma G\rho}}$

QUESTION 4 [25]

a) Given the relation $\frac{dP}{dr} = -G\frac{M_r\rho}{r^2}$, the stellar boundary condition $P = 0$ when $r = R$ and assuming constant density,

i) Show that $P(r) = \frac{2}{3}\pi G\rho^2\left(R^2 - r^2\right)$ [derivation] (5)

ii) The pressure and temperature at the centre of a white dwarf of radius $R = 6.37\times10^6$ m, constant density $\rho = 1.84\times10^9$ kg.m^{-3} and a carbon ($\mu = 12.000$) core. [assignment] (5)

i) $\frac{dP}{dr} = -G\frac{M_r\rho}{r^2} = ...$, … (after some calculations) … $\Rightarrow P(r) = \frac{2}{3}\pi G\rho^2\left(R^2 - r^2\right)$
ii) $P(0) = \frac{2}{3}\pi G\rho^2 R^2 = ...$, … (after some calculations) … $\Rightarrow T = 1.52\times10^{10}$ K

b) One way to imagine a neutron star is as a huge atomic nucleus with ~10^{57} nucleons. What would be its radius under this simplistic model? [class example] (3)

The formula for radius of an atomic nucleus … (after some calculations) … $R = 12$ km

c) Show that the minimum rotational period of a pulsar is $P_{\min} = 2\pi\sqrt{\frac{R^3}{GM}}$.

[derivation] (4)

… see notes

d) In no more than two sentences each, describe the following terms:

i) The Schwarzschild radius

ii) An interacting binary

iii) An accretion disk

iv) A supernova [conceptual] (8)

Any correct description of these terms (see notes, textbook, online, etc.)